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▶ **To cite this version:**

Mickaël dos Santos, Petru Mironescu, Oleksandr Misiats. The Ginzburg-Landau functional with a discontinuous and rapidly oscillating pinning term. Part I: the zero degree case. Commun. Contemp. Math., 2011, 13 (5), pp.885-914. hal-00515893

HAL Id: hal-00515893
https://hal.science/hal-00515893

Submitted on 8 Sep 2010

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The Ginzburg-Landau functional with a discontinuous and rapidly oscillating pinning term. Part I: the zero degree case

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Abstract

We consider minimizers of the Ginzburg-Landau energy with pinning term and zero degree Dirichlet boundary condition. Without any assumptions on the pinning term, we prove that these minimizers do not develop vortices in the limit $\varepsilon \to 0$. We next consider the specific case of a periodic discontinuous pinning term taking two values. In this setting, we determine the asymptotic behavior of the minimizers as $\varepsilon \to 0$.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain and let $a_\varepsilon : \Omega \to \mathbb{R}$ be a measurable function such that $0 < b \leq a_\varepsilon \leq 1$. We associate with a_ε a generalized Ginzburg-Landau (GL, in short) type energy

$$E_\varepsilon(u) = \frac{1}{2} \int_\Omega \left\{ |\nabla u(x)|^2 + \frac{1}{2\varepsilon^2}(a_\varepsilon^2(x) - |u(x)|^2)^2 \right\} dx. \tag{1}$$

Here, $u \in H^1(\Omega, \mathbb{C})$ and $\varepsilon > 0$ is the GL parameter.

This variant of the standard GL type energy (which corresponds to $a_\varepsilon \equiv 1$) is called GL functional with pinning term a_ε or pinned GL functional. We quote here few relevant papers among the vast literature concerning this energy functional.

- In [2], the authors consider the case where $a_\varepsilon = a \in C^\beta(\Omega)$ is independent of ε.

- [18] and [3] treat the case where $a_\varepsilon = a$ is independent of ε and takes the value b in ω and 1 outside ω, with ω smooth subset of Ω. The latter article considers the case of an applied magnetic field.

- In [1], a_ε depends on ε and is smooth. The oscillation rate of a_ε depends on ε.

The goal of this article is to study the pinned GL functional with a fast oscillating discontinuous pinning term a_ε. This may be viewed as a simplification of more realistic models which describe superconductivity phenomena for composite superconductors. The experimental pictures suggest nearly 2D structure of parallel vortex tubes ([21], Fig I.4). Therefore, the domain Ω can be viewed as a cross-section of a multifilamentary wire with a number of thin superconducting filaments. Such multifilamentary wires are widely used in industry, including magnetic energy-storing devices, transformers and power generators [17], [15].

Another important practical issue in modeling superconductivity is to decrease the energy dissipation in superconductors. Here, the dissipation occurs due to currents associated with the motion of vortices ([19], [4]). This dissipation as well the thermomagnetic stability can be improved by *pinning* ("fixing the positions") of vortices. This, in turn, can be done by introducing impurities or inclusions in the superconductor.

Our pinning term is periodic with respect to a $\delta \times \delta$ grid where $\delta = \delta(\varepsilon) \to 0$. As in [1], due to the fast oscillations, this problem is related to a periodic homogenization problem (depending on the relation between ε and δ).

The boundary condition we consider is the Dirichlet one. More specifically, we fix some $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$. Our class of test functions is

$$H^1_g := \{u \in H^1(\Omega, \mathbb{C}) \,|\, u = g \text{ on } \partial\Omega\}. \tag{2}$$

We consider solutions u_ε of the minimization problem

$$\inf_{u \in H^1_g} E_\varepsilon(u). \tag{3}$$

In this article we will consider only the case where the boundary data g has zero degree. The case where the degree is not zero requires additional techniques and will be investigated in a forthcoming paper.

Recall that the degree (winding number) of g is defined as

$$\deg_{\partial\Omega}(g) = \frac{1}{2\pi} \int_{\partial\Omega} g \times \partial_\tau g \,\mathrm{d}\tau = 0,$$

where:

- For $z \in \mathbb{C}$, $\Re z$ denotes the real part of z and $\Im z$ denotes the imaginary part of z.

- "$\times$" stands for the "vectorial product" in $\mathbb{C}$, $z_1 \times z_2 = \Im(\overline{z_1} z_2)$, $z_1, z_2 \in \mathbb{C}$.

- τ is the unit and direct tangent vector at $\partial\Omega$, *i.e.*, denoting ν to be the unit outward normal to $\partial\Omega$, one has $\tau = \nu^\perp$.

- ∂_τ is the tangential derivative.

This degree is an integer. For a proof of this assertion and for more properties of the topological degree of g, see *e.g.* [10] or [5].

If u_ε is a minimizer of the problem (3), then it satisfies the Euler-Lagrange equation

$$\begin{cases} -\Delta u_\varepsilon = \dfrac{1}{\varepsilon^2} u_\varepsilon (a_\varepsilon^2 - |u_\varepsilon|^2) & \text{in } \Omega \\ u_\varepsilon = g & \text{on } \partial\Omega \end{cases}. \tag{4}$$

Following [18], one may prove that in the special case $g \equiv 1$ there is a unique minimizer U_ε. Moreover, this minimizer satisfies $b \leq U_\varepsilon \leq 1$. This U_ε plays an important role in the study of GL functional with pinning term. Indeed, define, for $u \in H^1_g$, a new map $v = \dfrac{u}{U_\varepsilon} \in H^1_g$. Then E_ε decouples as follows [18]

$$E_\varepsilon(u) = E_\varepsilon(U_\varepsilon v) = f(\varepsilon) + F_\varepsilon(v), \tag{5}$$

where

$$f(\varepsilon) := E_\varepsilon(U_\varepsilon), \quad F_\varepsilon(v) := \frac{1}{2}\int_\Omega \left\{ U_\varepsilon^2 |\nabla v|^2 + \frac{1}{2\varepsilon^2} U_\varepsilon^4 (1 - |v|^2)^2 \right\}. \tag{6}$$

Therefore, u minimizes E_ε in $H^1_g(\Omega)$ if and only if v minimizes F_ε in H^1_g. In what follows, we denote by v_ε a minimizer of F_ε in H^1_g.

Following again [18], we have $|v_\varepsilon| \leq 1$ and $|u_\varepsilon| \leq 1$ in Ω.

From (5) and (6) we see that the study of the pinned GL is reduced to the study of the weighted GL functional F_ε and to the study of the asymptotics of U_ε.

The plan of our work is the following: in Section 2 we prove a "clearing out" result (Theorem 1). More specifically, we prove that v_ε is "vortexless" for small ε, *i. e.*, that $|v_\varepsilon| \to 1$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$. (Recall that $\deg_{\partial\Omega}(g) = 0$; this assumption is essential for our conclusion.) This result is true for any weighted GL functionals. Such general functionals are defined by formula (7) and do not require any assumption except uniform bounds on the weights. In particular, clearing out does not rely on any periodicity assumption. We believe that this result has its own interest.
The clearing out result reduces the study of the behavior of v_ε to the one of $\mathbb{S}^1$-valued maps. In other words, we will reduce the problem of minimizing F_ε in the class of all test functions to the one of minimizing F_ε in the class of $\mathbb{S}^1$-valued maps. The latter problem will be studied in detail in Section 3. There, the asymptotic analysis of minimizers of the F_ε among $\mathbb{S}^1$-valued maps, combined with an asymptotic analysis of U_ε (analysis performed at the beginning of Section 3), will allow us to conclude Section 3 by describing the behavior of u_ε as $\varepsilon \to 0$.

Acknowledgments

The authors thank Leonid V. Berlyand and Georges Griso for useful and stimulating discussions. Part of this work was done while the first author was visiting the Pennsylvania State University. He thanks the Mathematics Department there for its hospitality. The work of M. Dos Santos and O. Misiats was partially supported by NSF grant DMS-0708324.

2 Clearing out for general weighted Ginzburg-Landau type functionals

Let $b \in (0,1)$ and let $\alpha_\varepsilon \in W^{1,\infty}(\Omega)$, $\beta_\varepsilon \in L^\infty(\Omega)$ be such that $b \le \alpha_\varepsilon, \beta_\varepsilon \le 1$. We associate to α_ε and β_ε the weighted GL type functional defined through the formula

$$\begin{array}{rccc} F_\varepsilon : & H^1(\Omega,\mathbb{C}) & \to & \mathbb{R}^+ \\ & v & \mapsto & \displaystyle\frac{1}{2}\int_\Omega \left\{\alpha_\varepsilon|\nabla v|^2 + \frac{\beta_\varepsilon}{2\varepsilon^2}(1-|v|^2)^2\right\} \end{array}. \tag{7}$$

Let $g \in H^{1/2}(\partial\Omega,\mathbb{S}^1)$ be such that $\deg_{\partial\Omega}(g) = 0$. For $\varepsilon > 0$, we denote by v_ε a minimizer of F_ε in H^1_g. One may easily prove that v_ε satisfies

$$\begin{cases} -\mathrm{div}(\alpha_\varepsilon\nabla v_\varepsilon) = \dfrac{\beta_\varepsilon}{\varepsilon^2}v_\varepsilon(1-|v_\varepsilon|^2) & \text{in } \Omega \\ v_\varepsilon = g & \text{on } \partial\Omega \end{cases}. \tag{8}$$

Since $\deg_{\partial\Omega}(g) = 0$, we have [7] $H^1_g(\Omega,\mathbb{S}^1) = \{v \in H^1_g \,|\, |v| = 1 \text{ in } \Omega\} \ne \emptyset$.
If we take any fixed map $v \in H^1_g(\Omega,\mathbb{S}^1)$ as a test function for F_ε, we find that there is C_0 depending only on g such that

$$\min_{v\in H^1_g(\Omega)} F_\varepsilon(v) = F_\varepsilon(v_\varepsilon) \le C_0. \tag{9}$$

2.1 Uniform convergence of $|v_\varepsilon|$ to 1

This part is devoted to the proof of the following theorem.

Theorem 1. *When $\varepsilon \to 0$, we have $|v_\varepsilon| \to 1$ uniformly in $\overline{\Omega}$.*

For the convenience of the reader, we split the rather long proof of Theorem 1 into two parts.

2.1.1 Theorem 1 holds far away the boundary

We prove that, for sufficiently small ε, $|v_\varepsilon|$ is arbitrarly close to 1 outside an $2\sqrt{\varepsilon}$-neighborhood of $\partial\Omega$.

Proposition 1. *Let $\varepsilon_n \downarrow 0$ and $\{x_n\}_n \subset \Omega$ be such that* $\mathrm{dist}(x_n,\partial\Omega) \ge 2\sqrt{\varepsilon}_n$. *Then $|v_{\varepsilon_n}(x_n)| \to 1$.*

Proof. We write ε instead of ε_n. Let n be sufficiently large such that $\sqrt{\varepsilon} > \varepsilon$ and consider the circular annulus $B_{\sqrt{\varepsilon}}(x_n) \setminus B_\varepsilon(x_n)$.

From (9), we have, with $\mathcal{C}_r := \{|x - x_n| = r\}$,

$$C_0 \ge \frac{b}{4}\int_{B_{\sqrt{\varepsilon}}(x_n)\setminus B_\varepsilon(x_n)} \left\{|\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\} = \frac{b}{4}\int_\varepsilon^{\sqrt{\varepsilon}} \frac{1}{r}\cdot r\int_{\mathcal{C}_r}\left\{|\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\}. \tag{10}$$

By mean value argument, there are C_1 (depending only on g, Ω and b) and $r \in (\varepsilon, \sqrt{\varepsilon})$ such that

$$r\int_{\mathcal{C}_r}\left\{|\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\} \le \frac{C_1}{|\ln\varepsilon|}. \tag{11}$$

Lemma 1. *Let $\delta > 0$. Then, for large n and for r as in (11), we have*

1. $\mathrm{Var}\,(v_\varepsilon, \mathcal{C}_r) \le \delta$, *where* $\mathrm{Var}\,(v_\varepsilon, \mathcal{C}_r) := \displaystyle\int_{\mathcal{C}_r} |\partial_\tau v_\varepsilon|$;

2. $|v_\varepsilon| \ge 1 - 2\delta$ *on* $\mathcal{C}_r$.

Proof. Assertion *1.* is a direct consequence of the bound (11), which yields

$$\left(\int_{\mathcal{C}_r} |\partial_\tau v_\varepsilon|\right)^2 \le \left(\int_{\mathcal{C}_r} |\nabla v_\varepsilon|\right)^2 \le \int_{\mathcal{C}_r} 1 \int_{\mathcal{C}_r} |\nabla v_\varepsilon|^2 = 2\pi r \int_{\mathcal{C}_r} |\nabla v_\varepsilon|^2 \le \frac{2\pi C_1}{|\ln \varepsilon|}.$$

It follows that, for large n, we have $|\mathrm{Var}\,(v_\varepsilon, \mathcal{C}_r)| \le \delta$.

In order to prove *2.*, we argue by contradiction. Assume that there are $\delta > 0$, a subsequence $\{n_k\}_k$ and points $x_{n_k} \in \mathcal{C}_r$ such that $|v_\varepsilon(x_{n_k})| < 1 - 2\delta$ (here $\varepsilon = \varepsilon_{n_k}$).

From the estimate *1.* on $\mathrm{Var}\,(v_\varepsilon, \mathcal{C}_r)$, one has, for large k, $|v_\varepsilon| < 1 - \delta$ on $\mathcal{C}_r$.

Consequently, $r \displaystyle\int_{\mathcal{C}_r} (1 - |v_{\varepsilon_{n_k}}|^2)^2 \ge 2\pi r^2 \delta^2$. Since $r \ge \varepsilon$, this inequality contradicts the estimate (11) for small ε. □

So far, we proved the existence of a circle around x_n such that, on that circle, $|v_\varepsilon|$ is close to 1 and v_ε varies little. More specifically: if $0 < \gamma < 1$ then, for large n, there exists $S_\varepsilon \subset \overline{B_1(0)}$ such that

- $\mathrm{dist}(S_\varepsilon, 0) \ge 1 - \gamma$,
- S_ε is the smallest of the two regions delimited by a chord in the closed unit disc,
- $v_\varepsilon(\mathcal{C}_r) \subset S_\varepsilon$.

The following lemma implies that, under the above assumptions on S_ε and on r, we have, for large n, $|v_\varepsilon(x_n)| \ge 1 - \gamma$. This inequality completes the proof of Proposition 1, which is the first step in the proof of Theorem 1. □

Lemma 2. *Let C be a chord in the closed unit disc, C different from a diameter. Let S be the smallest of the two regions enclosed by the chord and the boundary of the disc.*
Let O be a Lipchitz bounded open set and let $g \in H^{1/2}(\partial O, S)$.
Let $\tilde\alpha, \tilde\beta \in L^\infty(O, \mathbb{R})$ satisfy $\operatorname{ess\,inf} \tilde\alpha > 0, \operatorname{ess\,inf} \tilde\beta > 0$.
If v minimizes GL type energy

$$\tilde F(v) = \int_O \left\{ \tilde\alpha(x) |\nabla v|^2 + \tilde\beta(x)(1 - |v|^2)^2 \right\}$$

in $H^1_g(O)$, then $v(O) \subset S$.

Remark 1. *This statement generalizes Lemma 8 in [6] (there $\tilde\alpha = 1, \tilde\beta = 1/(2\varepsilon^2)$). However, the proof in [6] does not apply directly to our situation.*

Proof. Clearly, one may assume that O is connected.
We start by noting that v has the following properties:

- v is continuous in O (this relies on the equation satisfied by v, on Theorem 2 in [20] and on Sobolev embeddings).
- $|v| \leq 1$. Indeed, consider the test function $v' = \begin{cases} v, & \text{if } |v| \leq 1 \\ \dfrac{v}{|v|}, & \text{if } |v| > 1 \end{cases}$. Since v' has more energy than v, we find that $|v| \leq 1$ a. e. and thus $|v| \leq 1$.

Without loss of generality, we may assume that, for some $\mu \in (0,1)$, we have $C = \{z \in B_1(0) : \Re z = \mu\}$ and $S = \{z \in \overline{B_1(0)}; \Re z \geq \mu\}$.

The map $w := |\Re v| + i\Im v$ equals g on ∂O and has the same energy as v. Thus w minimizes $\tilde{F}$. In particular, w is continuous. Therefore, if we prove that $w(O) \subset S$, we will have $v(O) \subset S$. In conclusion, we reduced the problem to the case where $\Re v \geq 0$.

Let P be the orthogonal projection on S. When $z \in B_1(0) \cap \{\Re z \geq 0\}$, we have

$$P(z) = \begin{cases} z, & \text{if } \Re z \geq \mu \\ \mu + i\Im z, & \text{if } |\Im z| \leq \sqrt{1-\mu^2} \text{ and } \Re z < \mu \\ \mu + i(\text{sign } \Im z)\sqrt{1-\mu^2}, & \text{if } |\Im z| > \sqrt{1-\mu^2} \text{ and } \Re z < \mu \end{cases}. \tag{12}$$

One may check easily that

$$|z| \leq |P(z)| \leq 1 \text{ for } z \in B_1(0) \cap \{\Re z \geq 0\}. \tag{13}$$

Set $\psi(z) := P(w(z))$, which equals g on ∂O. Since P is 1-Lipschitz, we have $|\nabla \psi| \leq |\nabla w|$. On the other hand, (13) implies $|w| \leq |\psi| \leq 1$. Consequently, $\tilde{F}(\psi) \leq \tilde{F}(w)$.

Since w is a minimizer, ψ is also a minimizer. Using the previous pointwise estimates and the equality of the energies, one may conclude that $|\psi(z)| = |w(z)|$ for each z (by continuity of w and ψ) and $|\nabla \psi| = |\nabla w|$ a.e.

By solving the equation $|z| = |P(z)|$, we see that $|\psi| = |w|$ implies that w takes values in $S \cup V$, where $V := \{z' \in \mathbb{S}^1 \,|\, 0 \leq \Re z' < \mu\}$.

We have to prove that $U := w^{-1}(V) = \emptyset$. We argue by contradiction and assume $U \neq \emptyset$. Then U is open, since $U = O \setminus w^{-1}(S)$ with S a closed set.

We first prove that w is locally constant in U. Indeed, in U, w satisfies $\operatorname{div}(\alpha \nabla w) = 0$. Since $w \in H^1(U, \mathbb{S}^1)$, we may write, in U, $w = e^{\imath\varphi}$, where $\varphi \in H^1$ [8]. Let $\zeta \in C_c^\infty(U)$. If we multiply the equation $\operatorname{div}(\alpha \nabla(\cos\varphi)) = 0$ by $\zeta \cos\varphi$ and the equation $\operatorname{div}(\alpha \nabla(\sin\varphi)) = 0$ by $\zeta \sin\varphi$ and add the two results, we obtain $\displaystyle\int \alpha\zeta|\nabla\varphi|^2 = 0$, so that φ (and thus w) is locally constant in U.

Let $W \neq \emptyset$ be a connected component of U, so that $w \equiv s \in V$ in W. Consider the non empty set $Y := w^{-1}(\{s\})$. Then Y is open in O (since w is locally constant in U), and clearly Y is closed in O. Therefore, $Y = O$, *i. e.*, $w \equiv s$ in O. This contradicts the facts that $g : \partial O \to S$, $\text{tr}_{\partial O} w = g$ and $s \notin S$. $\square$

2.1.2 Theorem 1 holds close to the boundary

We prove that, inside an $o_\varepsilon(1)$-strip along $\partial\Omega$ and for sufficiently small ε, $|v_\varepsilon|$ is arbitrarily close to 1.

The key argument will be provided by the following lemma.

Lemma 3. *Let $(x_\varepsilon)_{\varepsilon>0} \subset \Omega$ be such that $r_\varepsilon := \operatorname{dist}(x_\varepsilon, \partial\Omega) \to 0$. Then we have, for all $C \geq 2$, $F_\varepsilon(v_\varepsilon, B_{Cr_\varepsilon}(x_\varepsilon)) \to 0$.*

Proof. Note that it suffices to prove the result for $C = 2$. (For larger values of C, it suffices to replace x_ε by the point at distance $\dfrac{C+1}{2}r_\varepsilon$ from x_ε and at distance $\dfrac{C+3}{2}r_\varepsilon$ from $\partial\Omega$.)

Let $\delta > 0$. We will prove that there is $\varepsilon_\delta > 0$ such that for $\varepsilon < \varepsilon_\delta$, we have $F_\varepsilon(v_\varepsilon, B_{2r_\varepsilon}(x_\varepsilon)) \leq \delta$. For the convenience of the reader, the proof is divided into four steps.

Step 1: Flattening of Ω and choice of a good triangle
Without loss of generality, we may assume that $\partial\Omega$ is flat near x_ε. The general case is obtained by flattening the boundary. This will affect the equation satisfied by v_ε and the energy associated with it, but not the conclusion of the proof below (which relies only on energy bounds and qualitative conclusions derived form the equation of v_ε). From now on, we assume that $\Omega \subset \mathbb{R}^2_+$ and $\partial\Omega \subset \mathbb{R}$ in a neighborhood of fixed size of x_ε. We also assume, without loss of generality, that $x_\varepsilon = (0, r_\varepsilon)$.

For $\ell > 0$, we set

$$T_\ell := \{(s,t)\,|\,t = s + \ell,\ s \in [-\ell, 0]\} \cup \{(s,t)\,|\,t = -s + \ell,\ s \in (0, \ell]\} \subset \mathbb{R}^2_+$$

(thus T_ℓ is the union of two segments).
Denote by ω_ℓ the (solid) triangle enclosed by T_ℓ and $\mathbb{R}$. Then we have $B(x_\varepsilon, 2r_\varepsilon) \cap \Omega \subset \omega_{5r_\varepsilon}$.
Our goal is to construct, for an appropriate small ℓ (depending on x_ε and such that $\ell > 5r_\varepsilon$) a test function $h : \omega_\ell \to \mathbb{C}$ such that $\operatorname{tr}_{\partial\omega_\ell} h = \operatorname{tr}_{\partial\omega_\ell} v_\varepsilon$ and $F_\varepsilon(h, \omega_\ell) \to 0$. Since v_ε is a global minimizer of F_ε in $H^1_g(\Omega, \mathbb{C})$, it follows that v_ε is also a minimizer of F_ε in $H^1_{\operatorname{tr}_{\partial\omega_\ell} v}(\omega_\ell, \mathbb{C})$.
Our goal is to prove that $F_\varepsilon(v_\varepsilon, \omega_\ell) \to 0$. Since $B_{2r_\varepsilon}(x_\varepsilon) \subset \omega_\ell$, the lemma will follow.

Let $\varepsilon_1 > 0$ be such that for $\varepsilon < \varepsilon_1$, $5r < \sqrt{r}$. Let w be the harmonic extension of g to Ω. We claim that

1. $\exists\, C_1 > 0$ (independent of ε) and $\exists\ \ell \in (5r, \sqrt{r})$ such that
$$\ell \int_{T_\ell} \left\{ |\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1 - |v_\varepsilon|^2)^2 + |\nabla w|^2 \right\} \leq \frac{C_1}{|\ln r|}, \tag{14}$$

2. $|v_\varepsilon(x)| \underset{x \in T_\ell,\, x \to \partial\Omega}{\longrightarrow} 1$,

3. $|v_\varepsilon| \geq 1/2$ on T_ℓ (for sufficiently small ε).

The claim 1. comes directly from (9) and a mean value argument.
Claim 2. is proved in Lemma 4 below, using an argument essentially due to Boutet de Monvel and Gabber [14].

In order to prove Claim 3., we start by noting that

$$\left(\int_{T_\ell} |\partial_\tau |v_\varepsilon||\right)^2 \le C\ell \int_{T_\ell} |\partial_\tau |v_\varepsilon||^2 \le C\ell \int_{T_\ell} |\nabla v_\varepsilon|^2 \le \frac{C'}{|\ln r|}. \tag{15}$$

Consequently, there exists $0 < \varepsilon_2 \le \varepsilon_1$ such that, for $\varepsilon < \varepsilon_2$, the variation of $|v_\varepsilon|$ on T_ℓ is smaller than 1/2. Since, by Lemma 4, we have $|v_\varepsilon| = 1$ at the endpoints of T_ℓ, we obtain that Claim 3. holds.

Lemma 4. *Let $\alpha \in W^{1,\infty}(\Omega)$, $\beta \in L^\infty(\Omega;\mathbb{R}_+)$ be such that $\inf \alpha > 0$. Let v be a critical point of $u \mapsto \int \alpha|\nabla u|^2 + \int \beta(1-|u|^2)^2$ in the class $H^1_g(\Omega)$, where $g \in H^{1/2}(\partial\Omega;\mathbb{S}^1)$. Then $|v| \in C(\overline{\Omega})$.*

Proof. We first note that $|v| \le 1$ a. e. (by the maximum principle. This is obtained, e. g., by noting that $U := 1 - |v|^2$ satisfies $\begin{cases} -\mathrm{div}(\alpha\nabla U) + 4\beta|v|^2 U = 2\alpha|\nabla v|^2 & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases}$, and consequently $U \ge 0$ in Ω.) We next split $v = v_1 + v_2$, where v_1 is the harmonic extension of g. It follows that v_2 satisfies $\begin{cases} -\Delta v_2 = \alpha^{-1}\nabla\alpha \cdot \nabla v + 2\alpha^{-1}\beta v(1-|v|^2) & \text{in } \Omega \\ v_2 = 0 & \text{on } \partial\Omega \end{cases}$. Since $|v| \le 1$ and $\alpha \in W^{1,\infty}$, we obtain $v_2 \in H^2(\Omega) \cap H^1_0 \subset C_0(\overline{\Omega})$. On the other hand, we have $v_1 \in C(\Omega)$ and $|v_1| \in C(\overline{\Omega})$ (the last point is essentially due to Boutet de Monvel and Gabber [14]; see also [11], Theorem A.3.2). Therefore, we have $|v| \in C(\overline{\Omega})$. □

Now that ℓ was properly chosen, we construct our test function h. This function will coincide with v_ε outside ω_ℓ. Therefore, we will only explain how to construct h inside ω_ℓ. In order to obtain a globally H^1-map, we will set h equal v_ε on T_ℓ. Let h be of the form $h = \rho e^{\imath\psi}$; in order to have $h = v_\varepsilon$ on T_ℓ, we will make sure that $\rho = |v|$ and $e^{\imath\psi} = \dfrac{v_\varepsilon}{|v_\varepsilon|}$ on T_ℓ. In Step 2, we construct ρ. In Step 3, we construct ψ. Finally, in Step 4 we estimate the energy of h and conclude.

Step 2 : Choice of the modulus ρ of the test function h

Let $\rho : \overline{\omega}_\ell \to [0,1]$ be defined by

$$\rho(s,t) = \begin{cases} \dfrac{t}{s+\ell}\left(|v_\varepsilon(s, s+\ell)| - 1\right) + 1, & \text{if } s < 0 \\ \dfrac{t}{-s+\ell}\left(|v_\varepsilon(s, -s+\ell)| - 1\right) + 1, & \text{if } s > 0 \end{cases}.$$

Clearly, $\rho \in H^1(\omega_\ell, [0,1])$, $\rho = |v_\varepsilon|$ on T_ℓ and $\rho = 1$ on $\partial\omega_\ell \cap \partial\Omega$.

For further use, we estimate $\int_{\omega_\ell} \left\{ |\nabla\rho|^2 + \frac{1}{\varepsilon^2}(1-\rho^2)^2 \right\}$. We denote $\omega_\ell^- = \{x = (s,t) \in \omega_\ell \,|\, s < 0\}$ (this is the left half of the triangle ω_ℓ). We will estimate the quantity $\int_{\omega_\ell^-} \left\{ |\nabla\rho|^2 + \frac{1}{\varepsilon^2}(1-\rho^2)^2 \right\}$. By symmetry, a similar estimate will hold in $\omega_\ell^+ := \omega_\ell \setminus \overline{\omega_\ell^-}$, and thus in ω_ℓ.

We have

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{\omega_\ell^-}(1-\rho^2)^2 \leq & \frac{4}{\varepsilon^2}\int_{\omega_\ell^-}(1-\rho)^2 \leq \frac{C}{\varepsilon^2}\int_{-\ell}^{0}\mathrm{d}s\int_0^{\ell+s}\frac{t^2}{(s+\ell)^2}\left(|v_\varepsilon(s,s+\ell)|-1\right)^2\mathrm{d}t \\ \leq & \frac{C\ell}{\varepsilon^2}\int_{-\ell}^{0}\left(|v_\varepsilon(s,s+\ell)|-1\right)^2\mathrm{d}s \leq \frac{C\ell}{\varepsilon^2}\int_{T_\ell}\left(|v_\varepsilon|-1\right)^2\mathrm{d}s \leq \frac{C}{|\ln r|}.\end{aligned}$$

(The last inequality comes from Claim 1.)

In order to estimate $\displaystyle\int_{\omega_\ell^-}|\nabla\rho|^2$, we start from the identity

$$\int_{\omega_\ell^-}|\nabla\rho|^2 = \int_{-\ell}^{0}\mathrm{d}s\int_0^{\ell+s}\mathrm{d}t\,\left\{|\partial_s\rho|^2+|\partial_t\rho|^2\right\}.$$

On the one hand,

$$\begin{aligned}\int_{-\ell}^{0}\mathrm{d}s\int_0^{\ell+s}\mathrm{d}t|\partial_t\rho|^2 = & \int_{-\ell}^{0}\frac{(|v_\varepsilon(s,s+\ell)|-1)^2}{s+\ell}\,\mathrm{d}s \quad = \int_{-\ell}^{0}\frac{\mathrm{d}s}{s+\ell}\left(\int_{-\ell}^{s}\frac{\mathrm{d}}{\mathrm{d}k}\left[|v_\varepsilon|(k,k+\ell)\right]\right)^2 \\ \leq & \sqrt{2}\ell\int_{T_\ell}|\nabla v_\varepsilon|^2 \leq \frac{C}{|\ln r|}.\end{aligned}$$

On the other hand, we have

$$|\partial_s\rho|^2 \leq 2\left(\frac{t^2}{(s+\ell)^4}(|v_\varepsilon(s,s+\ell)|-1)^2+\frac{t^2}{(s+\ell)^2}(\nabla|v_\varepsilon|(s,s+\ell)\cdot(1,1))^2\right) = 2(A_1+A_2).$$

Since

$$\int_{-\ell}^{0}\int_0^{\ell+s}A_1 \leq \int_{-\ell}^{0}\int_0^{\ell+s}\frac{1}{(s+\ell)^2}(|v_\varepsilon(s,s+\ell)|-1)^2 = \int_{-\ell}^{0}\frac{1}{s+\ell}(|v_\varepsilon(s,s+\ell)|-1)^2 \leq \frac{C}{|\ln r|}$$

and

$$\int_{-\ell}^{0}\int_0^{\ell+s}A_2 \leq 2\ell\int_{T_\ell}|\nabla v_\varepsilon|^2 \leq \frac{C}{|\ln r|},$$

we find that $\displaystyle\int_{\omega_\ell}|\nabla\rho|^2 \leq \frac{C}{|\ln r|}$. In conclusion, the following estimate holds:

$$\int_{\omega_\ell}\left\{|\nabla\rho|^2+\frac{1}{\varepsilon^2}(1-\rho^2)^2\right\} \leq \frac{C}{|\ln r|}. \tag{16}$$

Step 3 : Construction of an auxiliary phase ψ

Recall that $|w(z)| \to 1$ uniformly as $z \to \partial\Omega$ [11]. Thus, there is some $0<\varepsilon_3\leq\varepsilon_2$ such that for $\varepsilon<\varepsilon_3$ we have $|w|\geq 1/2$ in ω_ℓ. For $\varepsilon<\varepsilon_3$, we may write, in ω_ℓ, $w=|w|\mathrm{e}^{\imath\varphi}$ with $\varphi\in H^1(\omega_\ell,\mathbb{R})$. Note that, by choice of ℓ, we have $|v_\varepsilon|\geq 1/2$ on T_ℓ and $v_\varepsilon\in H^1(T_\ell)$. Therefore, we may write $v_\varepsilon=|v_\varepsilon|\mathrm{e}^{\imath\phi}$ on T_ℓ, with $1/2\leq|v_\varepsilon|\leq 1$ and $\phi\in H^1(T_\ell)$.

Since $v_\varepsilon - w \in C(\overline{\Omega})$ (*cf* the proof of Lemma 4) and $v_\varepsilon = w$ on $\partial\Omega$, it follows that $\lim\limits_{z\to\partial\Omega}(v_\varepsilon - w)(z) = 0$. Therefore, we have

$$\lim_{\substack{z\to\partial\Omega\\ z\in T_\ell\cap\partial\omega_\ell^-}} \mathrm{e}^{\imath(\phi(z)-\varphi(z))} = \lim_{\substack{z\to\partial\Omega\\ z\in T_\ell\cap\partial\omega_\ell^+}} \mathrm{e}^{\imath(\phi(z)-\varphi(z))} = 1.$$

Consequently, there are $k_+, k_- \in \mathbb{Z}$ such that

$$\lim_{\substack{z\to\partial\Omega\\ z\in T_\ell\cap\partial\omega_\ell^-}} \frac{\phi(z)-\varphi(z)}{2\pi} = k_- \text{ and } \lim_{\substack{z\to\partial\Omega\\ z\in T_\ell\cap\partial\omega_\ell^+}} \frac{\phi(z)-\varphi(z)}{2\pi} = k_+.$$

By (14) and the fact that $|v_\varepsilon|, |w| \geq \frac{1}{2}$ on T_ℓ, we obtain $\ell \int_{T_\ell} \left\{|\nabla\phi|^2 + |\nabla\varphi|^2\right\} \leq \frac{C}{|\ln r|}$.
Thus, for small ε, the variations of ϕ and φ are small on $\partial\omega_\ell \backslash \partial\Omega$ and consequently, there is $0 < \varepsilon_4 < \varepsilon_3$ such that for $\varepsilon < \varepsilon_4$, we have $k_- = k_+$. Without loss of generality, we may assume $k_- = k_+ = 0$.

Let $\psi : \overline{\omega}_\ell \to \mathbb{R}$ be defined by

1. $\mathrm{tr}_{\partial\omega_\ell}\psi = \mathrm{tr}_{\partial\omega_\ell}(\phi - \varphi)$,
2. $\psi(s,t) = \begin{cases} \dfrac{t}{\ell+s}\left[\phi(s,s+\ell) - \varphi(s,s+\ell)\right], & \text{if } s<0 \\ \dfrac{t}{\ell-s}\left[\phi(s,-s+\ell) - \varphi'(s,-s+\ell)\right], & \text{if } s>0 \end{cases}.$

For further use, we estimate the Dirichlet energy of ψ. It suffices to estimate the energy in ω_ℓ^-; a similar estimate holds in ω_ℓ.

We have

$$\int_{\omega_\ell^-} |\nabla\psi|^2 = \int_{-\ell}^0 \mathrm{d}s \int_0^{\ell+s} \mathrm{d}t \left\{|\partial_s\psi|^2 + |\partial_t\psi|^2\right\} = B_1 + B_2.$$

First, we obtain, denoting $\xi = \phi - \varphi$,

$$B_1 = \int_{-\ell}^0 \int_0^{\ell+s} |\partial_s\psi|^2 \quad\leq\quad 2\int_{-\ell}^0 \int_0^{\ell+s} \left\{\left|\frac{\xi(s,s+\ell)}{\ell+s}\right|^2 + \left|\frac{\mathrm{d}}{\mathrm{d}s}\xi(s,s+\ell)\right|^2\right\} = 2(B_{11} + B_{12}).$$

Now

$$\begin{aligned} B_{11} = &\int_{-\ell}^0 \frac{1}{\ell+s}|\xi(s,s+\ell)|^2 \leq \int_{-\ell}^0 \frac{1}{\ell+s}\left|\int_{-\ell}^s \left|\frac{\mathrm{d}}{\mathrm{d}\alpha}\xi(\alpha,\alpha+\ell)\right| \mathrm{d}\alpha\right|^2 \\ \leq &C\int_{-\ell}^0 \int_{T_\ell} |\mathrm{d}\xi|^2 \leq \ell \int_{T_\ell} |\mathrm{d}\xi|^2 \leq \frac{C}{|\ln r|}. \end{aligned}$$

Next, we have

$$B_{12} = \int_{-\ell}^0 \int_0^{\ell+s} |\mathrm{d}\xi|^2(s,s+\ell) \leq \ell \int_{T_\ell} |\mathrm{d}\xi|^2 \leq \frac{C}{|\ln r|}.$$

Similarly, we have $B_2 \leq \frac{C}{|\ln r|}$.
Finally, we find that

$$\int_{\omega_\ell} |\nabla\psi|^2 \leq \frac{C}{|\ln r|}. \tag{17}$$

Step 4: Conclusion (proof of Lemma 3 completed)

Consider the following test function

$$h := \begin{cases} v & \text{in}\, \Omega \setminus \omega_\ell \\ \rho e^{\imath(\varphi+\psi)} & \text{in}\, \omega_\ell \end{cases}.$$

Clearly $h \in H^1_g$ and

$$F_\varepsilon(v_\varepsilon, B_{2r_\varepsilon}(x_\varepsilon)) \le F_\varepsilon(v_\varepsilon, \omega_\ell) \le F_\varepsilon(h, \omega_\ell) \le \frac{C}{|\ln r|} + 4\int_{\omega_\ell} |\nabla w|^2. \tag{18}$$

The last estimate follows by combining (16) with (17) and the fact that $|\nabla h|^2 = |\nabla \rho|^2 + \rho^2 |\nabla(\varphi+\psi)|^2$.

Since $\displaystyle\int_{\omega_\ell} |\nabla w|^2 \to 0$ as $\varepsilon \to 0$, we find that $F_\varepsilon(v, B_{2r}(x)) < \delta$ for small ε. □

The next result completes the proof of Theorem 1.

Proposition 2. *Let $\varepsilon_n \downarrow 0$ and $\{x_n\}_n \subset \Omega$ be such that* $\mathrm{dist}(x_n, \partial\Omega) \to 0$. *Then* $|v_{\varepsilon_n}(x_n)| \to 1$.

Proof of Proposition 2. Let $\delta \in (0,1)$. Denote $d_n := \mathrm{dist}(x_n, \partial\Omega)$ and $v_n := v_{\varepsilon_n}$. Since there is $C_0 > 0$ such that $F_{\varepsilon_n}(v_n) \le C_0$, we may choose $C_1 > 1$ and $r_n \in (d_n/C_1, d_n)$ such that

$$\frac{2\pi C_0}{\ln C_1} < \frac{\delta}{10^4} \tag{19}$$

and

$$r_n \int_{\mathcal{C}_n} \left\{ |\nabla v_n|^2 + \frac{1}{\varepsilon_n^2}(1 - |v_n|^2)^2 \right\} \le \frac{C_0}{\ln C_1}, \text{ with } \mathcal{C}_n = \{x \in \Omega \,|\, |x - x_n| = r_n\}. \tag{20}$$

As in the proof of *1.* in Lemma 1, we have

$$[\mathrm{Var}(v_n, \mathcal{C}_n)]^2 \le \frac{2\pi C_0}{\ln C_1}. \tag{21}$$

Using (21) and the bound (19), we find that one of the two cases occurs:

1. $|v_n| \ge 1 - \dfrac{\delta}{10}$ on $\mathcal{C}_n$,
2. $|v_n| < 1 - \dfrac{\delta}{10^3}$ on $\mathcal{C}_n$.

In the first case, using (21) and Lemma 2, we obtain $|v_n(x_n)| \ge 1 - \delta$.

Assume that for infinitely many n the second case occurs. Up to subsequence, we may assume that it is true for each n.

For large n, let $y_n := \Pi_{\partial\Omega}(x_n)$ be the orthogonal projection of x_n on $\partial\Omega$ and let x_n' be the intersection point of the segment $[x_n, y_n]$ with $\mathcal{C}_n$. For large n and for all $z \in T_n := \left\{ z \in \mathcal{C}_n \,|\, |x_n' - z| \le \dfrac{r}{2} \right\}$ we have

$$|z - w_z| \le 3d_n. \tag{22}$$

Here, w_z is the first intersection point with $\partial\Omega$ of the ray starting from x and passing through z.

Note that
$$z \in T_n \Leftrightarrow z = x_n + (x_n' - x_n)e^{\imath\theta} \text{ with } \theta \in [-\pi/6, \pi/6]. \tag{23}$$

For $\theta \in [-\pi/6, \pi/6]$ we denote $I_\theta := [z, w_z]$, where $z = z(\theta)$ is given by (23). Since $|v_n(z)| < 1 - \dfrac{\delta}{10^3}$ and $|v_n(w_z)| = 1$ we have
$$\frac{\delta^2}{10^6} \le \left(\int_{I_\theta} \partial_\tau |v_n|\right)^2 \le 3d_n \int_{I_\theta} |\partial_\tau v_n|^2. \tag{24}$$

Denote $A := \displaystyle\bigcup_{\theta\in[-\pi/6,\pi/6]} I_\theta$ and write each $x \in A$ as $x = x_n + s e^{\imath\theta}$ $(s \ge r_n)$. By (22), (23) and (24) we have
$$\int_A |\nabla v_n|^2 \ge \int_{-\pi/6}^{\pi/6} \mathrm{d}\theta \int_{I_\theta} |\partial_\tau v_n|^2 s \,\mathrm{d}s \ge \frac{\pi}{3C_1} \inf_{z\in T_n} d_n \int_{I_\theta} |\partial_\tau v_n|^2 \ge \frac{\pi\delta^2}{9\cdot 10^6 \cdot C_1}.$$

Since C_1 is independent of n and $A \subset B_{3d_n}(x_n)$, the above estimate contradicts Lemma 3.

Hence, for sufficiently large n, we have $|v_n| \ge 1 - \dfrac{\delta}{10}$ on $\mathcal{C}_n$. This estimate together with Lemma 2 implies $|v_n(x_n)| \ge 1 - \delta$. □

2.2 A corollary of Theorem 1

From Theorem 1 one may easily prove that the contribution of the modulus is negligible. Indeed we have

Corollary 1. *The following hold.*

1. *We have* $\displaystyle\int_\Omega \left\{ |\nabla|v_\varepsilon||^2 + \frac{1}{\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \right\} \to 0$ *as* $\varepsilon \to 0$.
   In particular, we have $|v_\varepsilon| \to 1$ *in* $H^1(\Omega)$.
2. *Assume that (possibly along some subsequence) we have* $\alpha_\varepsilon \to \kappa$ *in* $L^2(\Omega)$. *Write* $g = e^{\imath\varphi_0}$ *(see [7]), where* $\varphi_0 \in H^{1/2}(\partial\Omega, \mathbb{R})$. *Write, for small* ε, $v_\varepsilon = |v_\varepsilon| e^{\imath\varphi_\varepsilon}$, *where* $\varphi_\varepsilon \in H^1_{\varphi_0}(\Omega, \mathbb{R})$. *Then* $\varphi_\varepsilon \to \varphi^*$ *in* $H^1(\Omega)$, *where* φ^* *is the solution of* $\begin{cases} -\mathrm{div}(\kappa\nabla\varphi^*) = 0 & \text{in } \Omega \\ \varphi^* = \varphi_0 & \text{on } \partial\Omega \end{cases}$.

The above statement implicitly uses two results on lifting, for which we refer to [8, 9]. The first one is that each zero degree map $g \in H^{1/2}(\partial\Omega; \mathbb{S}^1)$ may be lifted as $g = e^{\imath\varphi_0}$ for some $\varphi_0 \in H^{1/2}(\partial\Omega; \mathbb{R})$. The second is that each map in $u \in H^1_g(\Omega; \mathbb{S}^1)$ may be written as $u = e^{\imath\varphi}$, with $\varphi \in H^1_{\varphi_0}(\Omega; \mathbb{R})$. Consequently, each map $u \in H^1_g(\Omega; \mathbb{R}^2)$ such that $0 < \mathit{essinf}\, |u| \le \mathit{esssup}\, |u| < \infty$ may be written as $u = \rho e^{\imath\varphi}$, where $\rho = |u| \in H^1_1(\Omega; \mathbb{R}_+)$ and $\varphi \in H^1_{\varphi_0}(\Omega; \mathbb{R})$.

Proof. We start by noting that $b \le \kappa \le 1$.

Let v_ε be a minimizer of F_ε in H^1_g. By Theorem 1, we may write, for small ε , $v_\varepsilon = \rho_\varepsilon e^{\imath\varphi_\varepsilon}$, with $1/2 \le \rho_\varepsilon := |v_\varepsilon| \le 1$ and $\varphi_\varepsilon \in H^1_{\varphi_0}(\Omega, \mathbb{R})$.

Recall that $F_\varepsilon(v_\varepsilon) \leq C_0$ (with C_0 depending only on g, Ω and b). Thus, for small ε, we have $\displaystyle\int_\Omega |\nabla\varphi_\varepsilon|^2 \leq \frac{8C_0}{b}$.

If we set $w_\varepsilon := \mathrm{e}^{\imath\varphi_\varepsilon} \in H^1_g$, then we have

$$F_\varepsilon(v_\varepsilon) = \frac{1}{2}\int_\Omega \left\{\alpha_\varepsilon(\rho_\varepsilon^2|\nabla\varphi_\varepsilon|^2 + |\nabla\rho_\varepsilon|^2) + \frac{\beta_\varepsilon}{2\varepsilon^2}(1-\rho_\varepsilon^2)^2\right\} \leq F_\varepsilon(w_\varepsilon) = \frac{1}{2}\int_\Omega \alpha_\varepsilon|\nabla\varphi_\varepsilon|^2.$$

Consequently,

$$\int_\Omega \left\{|\nabla\rho_\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-\rho_\varepsilon^2)^2\right\} \leq \frac{2}{b}\int_\Omega (1-\rho_\varepsilon^2)|\nabla\varphi_\varepsilon|^2 \leq \frac{16C_0}{b^2}\|1-\rho_\varepsilon^2\|_{L^\infty(\Omega)} \underset{\varepsilon\to 0}{\to} 0.$$

We now prove *2*. We start by noting that $\varphi_\varepsilon - \varphi^*$ satisfies

$$\begin{cases} -\mathrm{div}[\alpha_\varepsilon\rho_\varepsilon^2\nabla(\varphi_\varepsilon - \varphi^*)] = \mathrm{div}[(\alpha_\varepsilon\rho_\varepsilon^2 - \kappa)\nabla\varphi^*] & \text{in } \Omega \\ \varphi_\varepsilon - \varphi^* = 0 & \text{on } \partial\Omega \end{cases}.$$

By the Lax-Milgram theorem, we find that

$$\|\nabla(\varphi_\varepsilon - \varphi^*)\|_{L^2} \leq C\|(\alpha_\varepsilon\rho_\varepsilon^2 - \kappa)\nabla\varphi^*\|_{L^2}. \tag{25}$$

We will next use the following simple fact: if $|f_n| \leq C$ and $f_n \to f$ in L^2 and if $g_n \to g$ in L^2, then $f_n g_n \to fg$ in L^2. This implies that $\alpha_\varepsilon\rho_\varepsilon^2 - \kappa \to 0$ in L^2 as $\varepsilon \to 0$. Finally, (25) implies that $\varphi_\varepsilon \to \varphi^*$ in H^1.

□

2.3 More on the convergence of v_ε

This part provides a more quantitative version of Theorem 1. Specifically, under some additional hypotheses on the boundary data g or on the behavior of the weight α_ε, we derive estimates on the rate of convergence of $|v_\varepsilon|$ to 1 or derive better convergence of the phase φ_ε of v_ε respectively.

In what follows, we assume that $g \in W^{1-1/q,q}(\partial\Omega, \mathbb{S}^1)$ for some $q > 2$. Let $\varphi_0 \in W^{1-1/q,q}(\partial\Omega, \mathbb{R})$ be such that $\mathrm{e}^{\imath\varphi_0} = g$ (for the existence of φ_0, see, e. g., [8]). For a fixed measurable function $\kappa : \Omega \to [b, 1]$, let $\varphi^* \in W^{1,q}(\Omega, \mathbb{R})$ be the solution of $\begin{cases} -\mathrm{div}(\kappa\nabla\varphi^*) = 0 & \text{in } \Omega \\ \varphi^* = \varphi_0 & \text{on } \partial\Omega \end{cases}$.

Proposition 3. *There is $p \in (2, q], \alpha \in (0, 1), C > 0$ (depending only on q, b, Ω and g) such that, for $0 < \varepsilon < 1$ and v_ε a minimizer of F_ε in H^1_g, we have*

1. *$\{v_\varepsilon\}$ is bounded in $W^{1,p}$ by a constant C which depends only on g, b and Ω.*
2. *$\{v_\varepsilon\}$ is relatively compact in $C^{0,\alpha}(\overline{\Omega})$.*
3. *$1 - |v_\varepsilon| \leq C\varepsilon^\gamma$ and $\displaystyle\int_\Omega \left\{|\nabla|v_\varepsilon||^2 + \frac{1}{\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\} \leq C\varepsilon^\gamma$ with $\gamma = \dfrac{2\alpha}{2+\alpha}$.*

4. *Furthermore, if (possibly after passing to a subsequence) we have $\alpha_\varepsilon \to \kappa$ in L^2, then we have $\varphi_\varepsilon \to \varphi^*$ in $W^{1,p}$.*
   Here, we write, for small ε and in virtue of Theorem 1, $v_\varepsilon = \rho_\varepsilon e^{\imath\varphi_\varepsilon}$, with $\varphi_\varepsilon \in H^1_{\varphi_0}$, $\rho_\varepsilon := |v_\varepsilon| \in [1/2, 1]$.

Proof. Let φ be any fixed $W^{1,q}$-extension of φ_0. Then $\varphi_\varepsilon - \varphi$ satisfies

$$\begin{cases} -\mathrm{div}\left[\alpha_\varepsilon\rho_\varepsilon^2\nabla(\varphi_\varepsilon - \varphi)\right] = \mathrm{div}(\alpha_\varepsilon\rho_\varepsilon^2\nabla\varphi) & \text{in } \Omega \\ \varphi_\varepsilon - \varphi = 0 & \text{on } \partial\Omega \end{cases}. \tag{26}$$

Since

$$\|\alpha_\varepsilon\rho_\varepsilon^2\nabla\varphi\|_{L^q(\Omega)} \le C,$$

it follows from Theorem 1 in [20] that there are $p_1 \in (2, q]$ and $C > 0$ (depending only on b and Ω) such that $\|\nabla(\varphi_\varepsilon - \varphi)\|_{L^{p_1}(\Omega)} \le C$. Thus $\{\varphi_\varepsilon\}$ is bounded in $W^{1,p_1}(\Omega)$.

We next prove that $\|1-\rho_\varepsilon\|_{L^{p_1/2}} \le C\varepsilon^2$. For this purpose, we start with the equation satisfied by ρ_ε:

$$\begin{cases} \mathrm{div}(\alpha_\varepsilon\nabla\rho_\varepsilon) + \dfrac{\beta_\varepsilon}{\varepsilon^2}\rho_\varepsilon(1-\rho_\varepsilon^2) = \alpha_\varepsilon\rho_\varepsilon|\nabla\varphi_\varepsilon|^2 & \text{in } \Omega \\ 1-\rho_\varepsilon = 0 & \text{on } \partial\Omega \end{cases}. \tag{27}$$

Let $\eta_\varepsilon := 1-\rho_\varepsilon$ and $p_1 > 2$ be as in the conclusion of Theorem 1 in [20]. Set $r := p_1/2$ and consider a sequence $\{\phi_k\} \subset C^\infty([0,1],[0,1])$ such that

$$\phi_k \text{ is nondecreasing, } \phi_k(0) = 0 \text{ and } \phi_k(s) \to |s|^{r-1} \text{ as } k \to \infty,\ \forall\, s \in [0,1].$$

Let $A_\varepsilon := \beta_\varepsilon\rho_\varepsilon(1+\rho_\varepsilon)$, which satisfies, for small ε, $3b/4 \le A_\varepsilon \le 2$. Set $B_\varepsilon := \alpha_\varepsilon\rho_\varepsilon|\nabla\varphi_\varepsilon|^2$, which is bounded in $L^{p_1/2}$. If we multiply (27) by $\phi_k(\eta_\varepsilon)$, we find that

$$\int_\Omega \alpha_\varepsilon|\nabla\eta_\varepsilon|^2\phi_k'(\eta_\varepsilon) + \frac{1}{\varepsilon^2}\int_\Omega A_\varepsilon\eta_\varepsilon\phi_k(\eta_\varepsilon) = \int_\Omega B_\varepsilon\phi_k(\eta_\varepsilon).$$

Consequently, we have

$$\int_\Omega \eta_\varepsilon\phi_k(\eta_\varepsilon) \le C\varepsilon^2\int_\Omega B_\varepsilon\phi_k(\eta_\varepsilon). \tag{28}$$

Note that, in (28), the constant C depends only on b. By letting $k \to \infty$, we obtain, with s being the conjugate exponent of r, that

$$\int_\Omega \eta_\varepsilon^r \le C\varepsilon^2\int_\Omega B_\varepsilon\eta_\varepsilon^{r-1} \le C\varepsilon^2\left(\int_\Omega \eta_\varepsilon^r\right)^{\frac{1}{s}}\|B_\varepsilon\|_{L^r}.$$

This implies that $\|1-\rho_\varepsilon\|_{L^{p_1/2}} \le C\varepsilon^2$ which we wanted to prove.

Going back to (27), we observe that η_ε satisfies $\mathrm{div}(\alpha_\varepsilon\nabla\eta_\varepsilon) = h_\varepsilon$, where h_ε is bounded in $L^{p_1/2}(\Omega)$. Using again [20], we find that there is some $p_2 > 2$ such that $\nabla\eta_\varepsilon$ is bounded in $L^{p_2}(\Omega)$.

It follows that v_ε is bounded in $W^{1,p}(\Omega)$, with $p := \min(p_1, p_2) > 2$.

We next prove that $|1-\rho_\varepsilon| \le C\varepsilon^\gamma$ and $\displaystyle\int_\Omega \left\{|\nabla|v_\varepsilon||^2 + \frac{1}{\varepsilon^2}(1-|v_\varepsilon|^2)^2\right\} \le C\varepsilon^\gamma$, where $\gamma := \dfrac{p-2}{p-1}$.

Indeed, let $\alpha := 1 - \frac{2}{p}$, so that v_ε is bounded in $C^\alpha(\overline{\Omega})$ and $\int_\Omega (1-\rho_\varepsilon) \le C\varepsilon^2$. Let $x_0 = x_0(\varepsilon)$ be a minimum point of ρ_ε in $\overline{\Omega}$. Since Ω is smooth, for $r > 0$ sufficiently small we have $|B_r(x_0) \cap \Omega| \ge Cr^2$. It follows that

$$C\varepsilon^2 \ge \int_{B_r(x_0)} (1-\rho_\varepsilon) \ge C(1-\rho_\varepsilon(x_0) - Cr^\alpha)r^2.$$

With $r := \varepsilon^{\frac{2}{\alpha+2}}$, we find that $1 - \rho_\varepsilon(x_0) = \sup_{\overline{\Omega}}\{1-\rho_\varepsilon\} \le C\varepsilon^\gamma$.

The above estimate together with the inequality $F_\varepsilon(v_\varepsilon) \le F_\varepsilon(e^{\imath\varphi_\varepsilon})$ yield the bound on $\nabla\rho_\varepsilon$:

$$\int_\Omega \left\{ \alpha_\varepsilon |\nabla\rho_\varepsilon|^2 + \frac{\beta_\varepsilon}{2\varepsilon^2}(1-\rho_\varepsilon^2)^2 \right\} \le \int_\Omega \alpha_\varepsilon (1-\rho_\varepsilon^2)|\nabla\varphi_\varepsilon|^2 \le C\varepsilon^\gamma.$$

Finally, *4.* follows from the equation

$$\begin{cases} -\mathrm{div}\left[\alpha_\varepsilon\rho_\varepsilon^2 \nabla(\varphi_\varepsilon - \varphi^*)\right] = \mathrm{div}\left[(\alpha_\varepsilon\rho_\varepsilon^2 - \kappa)\nabla\varphi^*\right] & \text{in } \Omega \\ \varphi_\varepsilon - \varphi^* = 0 & \text{on } \partial\Omega \end{cases}.$$

Indeed, since $(\alpha_\varepsilon\rho_\varepsilon^2 - \kappa)\nabla\varphi^* \to 0$ in $L^{p_3}(\Omega)$ for a suitable p_3 such that $\nabla\varphi^* \in L^{p_3}$, we obtain, using again [20], that $\varphi_\varepsilon \to \varphi^*$ in W^{1,p_4}, for a suitable $p_4 > 2$. We conclude by choosing $p := \min\{p_1, \dots, p_4\}$. □

3 The Ginzburg-Landau functional with a periodic pinning term

In this part, we apply the results obtained in the previous section to the study of a GL energy with a discontinuous periodic pinning term. Inside unit square $Y = [0,1)^2$, consider a smooth subset $\omega \prec Y$, which will play a role of inclusion (or impurity). The relative size of this inclusion (with respect to the size of the square) will be controlled by some parameter $\lambda > 0$ in the following way: for $x_0 \in \omega$, we set $\omega_\lambda = \lambda\omega + (1-\lambda)x_0$. We now define the pinning term $a = a(x,\lambda)$ so that it takes different constant values inside and outside of the inclusion:

$$a(x,\lambda) = \begin{cases} b, & \text{if } x \in \omega_\lambda \\ 1, & \text{if } x \in Y \setminus \omega_\lambda \end{cases}, \tag{29}$$

where $b \in (0,1)$ is a fixed (material) parameter. We extend a to a periodic function in $\mathbb{R}^2$.

The analysis we develop here could apply to the more complicated situation where x_0 is allowed to depend on λ; however, we will not pursue in this direction here.

Let $\Omega \subset \mathbb{C}$ be a smooth, bounded, simply connected domain. For $1 > \delta > 0$, denote $\{C_n^\delta, n \ge 1\}$ a partition of $\mathbb{R}^2$ into squares with side δ; for simplicity, we suppose that the origin is an edge of one of the squares. We may assume, with no loss of generality, that the squares that lie inside Ω are labelled C_n^δ with $1 \le n \le N_\delta$. Denote $\Omega_\delta := \bigcup_{n=1}^{N_\delta} C_n^\delta$.

We define the pinning term in Ω as

$$a_\varepsilon(x) = \begin{cases} a(x/\delta, \lambda), & \text{if } x \in \Omega_\delta \\ 1, & \text{if } x \in \Omega \setminus \Omega_\delta \end{cases};$$

the notation a_ε is justified by the fact that we will later let δ depend on the GL parameter ε. The following energy will be associated with this pinning term:

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \left\{ |\nabla u|^2 + \frac{1}{2\varepsilon^2}(a_\varepsilon^2 - |u|^2)^2 \right\}.$$

Under the Dirichlet condition $\mathrm{tr}_{\partial\Omega} u = 1$, one has the existence of the unique minimizer U_ε of E_ε [18].

The following lemma is straightforward.

Lemma 5. *There exists a constant C (independent of $\varepsilon \in (0,1)$) such that*

$$E_\varepsilon(U_\varepsilon) \le C\lambda \min\left(\frac{1}{\varepsilon\delta}, \frac{\lambda}{\varepsilon^2}\right)$$

and

$$|\nabla U_\varepsilon| \le \frac{C}{\varepsilon}.$$

When $\varepsilon < \lambda\delta$, the above lemma is obtained by considering as a test function an ε-regularization of a_ε. When $\varepsilon \ge \lambda\delta$, it suffices to estimate the energy of the test function 1.

As explained in [18], if u is of modulus 1 on $\partial\Omega$ and we set $v := u/U_\varepsilon$, then the energy E_ε decouples as follows:

$$E_\varepsilon(u) = E_\varepsilon(U_\varepsilon) + F_\varepsilon(v),$$

where

$$F_\varepsilon(v) := \frac{1}{2}\int_\Omega \left\{ U_\varepsilon^2|\nabla v|^2 + \frac{U_\varepsilon^4}{2\varepsilon^2}(1 - |v|^2)^2 \right\}.$$

We next note that, by the maximum principle, we have $b \le U_\varepsilon \le 1$. Thus F_ε satisfies the assumptions of Theorem 1, Corollary 1 and Proposition 3. Therefore, if we let u_ε minimize E_ε in H^1_g, where $g : \partial\Omega \to \mathbb{S}^1$ is of zero degree, if U_ε minimizes E_ε in H^1_1 and if we decompose $u_\varepsilon = U_\varepsilon v_\varepsilon$, then the conclusions of these results apply to v_ε.

To be more specific, we fix $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$ such that $\deg_{\partial\Omega}(g) = 0$. Then:

1. there is some $\varphi_0 \in H^{1/2}(\Omega, \mathbb{R})$ is such that $g = e^{\imath\varphi_0}$
2. we decompose a minimizer u_ε of E_ε in H^1_g as $u_\varepsilon = U_\varepsilon v_\varepsilon$, where U_ε minimizes E_ε in H^1_1 and v_ε minimizes F_ε in H^1_g
3. using Theorem 1 we have, for small ε, $|u_\varepsilon| \ge b/2$. Thus we may decompose, for small ε, $u_\varepsilon = |u_\varepsilon| \mathrm{e}^{\imath\varphi_\varepsilon}$ with $\varphi_\varepsilon \in H^1_{\varphi_0}(\Omega, \mathbb{R})$
4. consequently, for small ε we have $v_\varepsilon = |v_\varepsilon| \mathrm{e}^{\imath\varphi_\varepsilon}$ with $|u_\varepsilon| = U_\varepsilon |v_\varepsilon|$.

From Corollary 1, we know that $|v_\varepsilon| \to 1$ uniformly and in H^1. Consequently, we will obtain the asymptotics of u_ε from the one of U_ε and of φ_ε.

The remaining part of this section is devoted to the asymptotic analysis of U_ε and v_ε; as a byproduct, this will give the asymptotics of u_ε. It turns out that the analysis is governed by the relation between ε and δ, as well as by the size of λ. Possibly after passing to subsequences and rescaling, we may assume, with no loss of generality, that we are in one of the four following cases:
Section 3.1: $\lambda \to 0$, the dilute case,
Section 3.2: $\lambda = 1, \delta = \varepsilon$, the critical case,
Section 3.3: $\lambda = 1, \varepsilon \ll \delta$, the physical case,
Section 3.4: $\lambda = 1, \delta \ll \varepsilon$, the non-physical case.

3.1 The dilute limit $\lambda \to 0$

3.1.1 Behavior of U_ε

In this case, the energy bound given by Lemma 5 immediately implies

Proposition 4. *We have*

$$U_\varepsilon \to 1 \text{ in } L^2(\Omega). \tag{30}$$

3.1.2 Limit of φ_ε

Proposition 5. *Let φ_* be the harmonic extension of φ_0 in Ω. Then, as $\varepsilon \to 0$,*

1. *$\varphi_\varepsilon \to \varphi_*$ in H^1*
2. *if, in addition, there is some $q > 2$ such that $g \in W^{1-1/q,q}(\partial\Omega)$, then we have $\varphi_\varepsilon \to \varphi_*$ in $W^{1,p}$ for some suitable $p \in (2, q]$.*

Proof. The first part is a direct consequence of Corollary 1 and of Proposition 4. The second part is a direct consequence of Propositions 3 and 4. □

3.2 The case $\lambda = 1, \delta = \varepsilon$

3.2.1 Limit of U_ε

Recall that $Y := [0, 1)^2$. Let

$$H^1_{\text{per}}(Y, \mathbb{R}) = \{u \in H^1(Y, \mathbb{R}) \,|\, \text{the extension by } Y\text{-periodicity of } u \text{ in } \mathbb{R}^2 \text{ is in } H^1_{\text{loc}}(\mathbb{R}^2)\}.$$

We define similarly $H^1_{\text{per}}(Y, \mathbb{C})$. For simplicity, we ignore the reference to $\mathbb{R}$ or $\mathbb{C}$ when irrelevant. Note that $u \in H^1(Y)$ extends to a Y-periodic H^1_{loc}-map if and only if

$$\text{tr}_{\{y_1=0\}} u(0, \cdot) = \text{tr}_{\{y_1=1\}} u(1, \cdot) \text{ and } \text{tr}_{\{y_2=0\}} u(\cdot, 0) = \text{tr}_{\{y_2=1\}} u(\cdot, 1)$$

$$\Leftrightarrow y_1(1 - y_1)\,[u(y_1, y_2) - u(y_1, 1 - y_2)] + y_2(1 - y_2)\,[u(y_1, y_2) - u(1 - y_1, y_2)] \in H^1_0(Y).$$

Using these characterizations of $H^1_{\text{per}}(Y)$, we find that $H^1_{\text{per}}(Y)$ is weakly H^1-closed. (For more properties of $H^1_{\text{per}}(Y)$, see, e. g., [13], part 3.4.)

It follows that there exists $\hat{u}$ which is a minimizer of

$$\mathcal{E}(u) = \frac{1}{2}\int_Y \left\{ |\nabla u|^2 + \frac{1}{2}(u^2 - a^2)^2 \right\} \text{ in the class } H^1_{\text{per}}(Y, \mathbb{R}).$$

Theorem 2. *The following hold:*

1. *The functional $\mathcal{E}$ has a unique (modulo multiplication by ± 1) minimizer $\hat{u}$ in $H^1_{per}(Y, \mathbb{R})$. Among the (exactly) two minimizers, one is positive, the other one negative*
2. *If $\hat{u}$ is the positive minimizer of $\mathcal{E}$ in $H^1_{per}(Y, \mathbb{R})$, then we have $U_\varepsilon \rightharpoonup \int_Y \hat{u}$ in $L^2(\Omega)$ as $\varepsilon \to 0$.*

Proof. We first investigate property *1.* This is done via the following two lemmas.

Lemma 6. *The energy functional $\mathcal{E}$ admits a positive global minimizer in $H^1_{per}(Y, \mathbb{R})$. Furthermore, all global minimizer have constant sign and satisfy*

$$-\Delta \hat{u} = \hat{u}(a^2 - \hat{u}^2) \text{ in } Y, \tag{31}$$

$$b \leq |\hat{u}| \leq 1, \tag{32}$$

$$\partial_\nu \hat{u}(0, y_2) = -\partial_\nu \hat{u}(1, y_2) \text{ and } \partial_\nu \hat{u}(y_1, 0) = -\partial_\nu \hat{u}(y_1, 1). \tag{33}$$

Proof. (31) is clear. In order to prove (32), let $u \in H^1_{\text{per}}(Y, \mathbb{R})$ minimize $\mathcal{E}$. Let

$$v := \begin{cases} |u|, & \text{if } b \leq |u| \leq 1 \\ 1, & \text{if } |u| > 1 \\ b, & \text{if } |u| < b \end{cases}.$$

It is clear that $v \in H^1_{\text{per}}(Y, \mathbb{R})$. On the other hand, we have

$$\mathcal{E}(v) = \frac{1}{2}\int_{\{b \leq |u| \leq 1\}} \left\{ |\nabla u|^2 + \frac{1}{2}(a^2 - u^2)^2 \right\} + \frac{1}{4}\int_{\{|u|>1\}} (a^2 - 1)^2 + \frac{1}{4}\int_{\{|u|<b\}} (a^2 - b^2)^2.$$

By the minimality of $\mathcal{E}(u)$, we find that $b \leq |u| \leq 1$ a. e. Noting that, if u is a minimizer, then u is continuous, we find that either u is either positive, or negative. In addition, either $b \leq u \leq 1$ or $-1 \leq u \leq -b$.

We next prove that minimizers $\hat{u}$ satisfy (33). Indeed, for all $\phi \in H^1_{\text{per}}(Y) \cap C(\overline{\Omega})$ we have

$$0 = \int_Y \nabla \hat{u} \cdot \nabla \phi - \hat{u}\phi(a^2 - \hat{u}^2) = -\int_{\partial Y} \phi \, \partial_\nu \hat{u}. \tag{34}$$

We next note that

$$\begin{aligned}
0 = \int_{\partial Y} \phi\ \partial_\nu \hat{u} &= \int_0^1 \left(\partial_\nu \hat{u}(0,t) + \partial_\nu \hat{u}(1,t)\right) \phi(0,t) + \int_0^1 \left(\partial_\nu \hat{u}(t,0) + \partial_\nu \hat{u}(t,1)\right) \phi(t,0) \\
&= T_1(\phi_1(t)) + T_2(\phi_2(t)),
\end{aligned}$$

with $\phi_1(t) = \phi(0,t)$ and $\phi_2(t) = \phi(t,0)$.

Since for each $\psi \in C_0^\infty((0,1),\mathbb{R})$ there is some $\phi \in H^1_{\text{per}}(Y,\mathbb{R})$ such that $\phi_1(t) = \psi(t)$ and $\phi_2 \equiv 0$, (34) implies that the map

$$\begin{array}{rccc}
T_1: & C_0^\infty((0,1),\mathbb{R}) & \to & \mathbb{R} \\
& \psi & \mapsto & \displaystyle\int_0^1 \left(\partial_\nu \hat{u}(0,t) + \partial_\nu \hat{u}(1,t)\right) \psi(t)
\end{array}$$

is identically zero. It follows that $\partial_\nu \hat{u}(0,t) + \partial_\nu \hat{u}(1,t) = 0$. A similar argument leads to $\partial_\nu \hat{u}(t,0) + \partial_\nu \hat{u}(t,1) = 0$. □

Lemma 7. *The energy $\mathcal{E}$ has a unique positive minimizer in $H^1_{per}(Y,\mathbb{R})$.*

Proof. Let u, v be two positive minimizers and let $w := v/u \in H^1_{\text{per}}$. By the energy decoupling formula [18] (which adapts to the periodic case), we have

$$E_\varepsilon(u) = E_\varepsilon(v) = E_\varepsilon(u) + \frac{1}{2}\int \left\{ u^2|\nabla w|^2 + \frac{1}{2}u^4(1-w^2)^2 \right\}.$$

Thus $w \equiv 1$, which implies $u = v$. □

As a next (and rather long) step in the proof of Theorem 2, we examine the asymptotic behavior of the energy carried by U_ε.

Proposition 6. *We have $\lim\limits_{\varepsilon\to} \varepsilon^2 E_\varepsilon(U_\varepsilon) = |\Omega|\mathcal{E}(\hat{u})$.*

Proof. We use the *unfolding operator* (see [12], definition 2.1). More specifically, we define, for $p \in (1,\infty)$,

$$\begin{array}{rccc}
\mathcal{T}_\varepsilon: & L^p(\Omega) & \to & L^p(\Omega \times Y) \\
& \phi & \mapsto & \mathcal{T}_\varepsilon(\phi)(x,y) = \left\{ \begin{array}{ll} \phi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right] + \varepsilon y\right), & \text{if } (x,y) \in \hat{\Omega}_\varepsilon \times Y, \\ 0 & \text{if } (x,y) \in \Lambda_\varepsilon \times Y \end{array} \right. ,
\end{array}$$

$$\hat{\Omega}_\varepsilon := \bigcup_{\substack{Y_\varepsilon^K \subset \Omega \\ Y_\varepsilon^K = \varepsilon(K+Y),\, K \in \mathbb{Z}^2}} \overline{Y_\varepsilon^K},\ \Lambda_\varepsilon := \Omega \setminus \hat{\Omega}_\varepsilon \text{ and } \left[\frac{x}{\varepsilon}\right] := \left(\left[\frac{x_1}{\varepsilon}\right], \left[\frac{x_2}{\varepsilon}\right]\right).$$

Here, for $s \in \mathbb{R}$, $[s]$ is the integer part of s.

We will use the following results:

i) $\mathcal{T}_\varepsilon$ is linear and continuous, of norm at most 1 ([12], prop. 2.5);

ii) $\mathcal{T}_\varepsilon(uv) = \mathcal{T}_\varepsilon(u)\mathcal{T}_\varepsilon(v)$ and $\mathcal{T}_\varepsilon\left(\dfrac{u}{v}\right) = \dfrac{\mathcal{T}_\varepsilon(u)}{\mathcal{T}_\varepsilon(v)}1\!\!1_{\hat{\Omega}_\varepsilon\times Y}$ ([12], equation (2.2));

iii) "Unfolding criterion for integrals" (u. c. i., [12], prop. 2.6) : If $\phi_\varepsilon \in L^1(\Omega)$ is such that $\displaystyle\int_{\Lambda_\varepsilon}|\phi_\varepsilon| \to 0$, then we have

$$\int_\Omega \phi_\varepsilon - \int_{\Omega\times Y}\mathcal{T}_\varepsilon(\phi) \to 0;$$

iv) $\varepsilon\mathcal{T}_\varepsilon(\nabla\phi)(x,y) = \nabla_y\mathcal{T}_\varepsilon(\phi)(x,y)$ for $\phi \in W^{1,p}(\Omega)$ ([12], equation (3.1)).

As a first step in the proof of Proposition 6, we prove that $\limsup_\varepsilon \varepsilon^2 E_\varepsilon(U_\varepsilon) \leq |\Omega|\mathcal{E}(\hat{u})$. Indeed, we consider the test function $H_\varepsilon \in H^1_1$ defined by

$$H_\varepsilon(x) := \rho_\varepsilon(x)\hat{u}\left(\left\{\frac{x}{\varepsilon}\right\}\right) + 1 - \rho_\varepsilon(x),$$

with

$$\rho_\varepsilon(x) := \min\left(1, \frac{\operatorname{dist}(x,\partial\Omega)}{\varepsilon}\right) \text{ and } \left\{\frac{x}{\varepsilon}\right\} = \frac{x}{\varepsilon} - \left[\frac{x}{\varepsilon}\right] \in Y.$$

Then we have

$$\mathcal{T}_\varepsilon(H_\varepsilon) \to \hat{u}(y) \text{ in } L^4(\Omega\times Y) \text{ and } \mathcal{T}_\varepsilon(\varepsilon\nabla H_\varepsilon)(x,y) \to \nabla_y\hat{u}(y) \text{ in } L^2(\Omega\times Y). \tag{35}$$

Indeed, the first convergence in (35) is a consequence of the fact that $\mathcal{T}_\varepsilon(H_\varepsilon) - \hat{u}(y)$ is bounded in $L^\infty(\Omega\times Y)$ and that its support is contained inside $\{x\in\Omega \,|\, \operatorname{dist}(x,\partial\Omega) < 3\varepsilon\}\times Y$. This implies at once that $\mathcal{T}_\varepsilon(H_\varepsilon) \to \hat{u}(y)$ in $L^4(\Omega\times Y)$.

In order to establish the second convergence in (35), we start from the identity

$$\begin{aligned}
\mathcal{T}_\varepsilon(\varepsilon\nabla H_\varepsilon) &= \mathcal{T}_\varepsilon(\rho_\varepsilon)\mathcal{T}_\varepsilon\left[\varepsilon\nabla\left(\hat{u}\left(\left\{\frac{x}{\varepsilon}\right\}\right)\right)\right] + \mathcal{T}_\varepsilon(\varepsilon\nabla\rho_\varepsilon)\mathcal{T}_\varepsilon\left[\hat{u}\left(\left\{\frac{x}{\varepsilon}\right\}\right) - 1\right] \\
&= \nabla_y\hat{u}(y)1\!\!1_{\hat{\Omega}_\varepsilon}(x) + (\mathcal{T}_\varepsilon(\rho_\varepsilon) - 1)\nabla_y\hat{u}(y)1\!\!1_{\hat{\Omega}_\varepsilon}(x) + \nabla_y\mathcal{T}_\varepsilon(\rho_\varepsilon)\mathcal{T}_\varepsilon\left[\hat{u}\left(\left\{\frac{x}{\varepsilon}\right\}\right) - 1\right] \\
&\equiv \nabla_y\hat{u}(y)1\!\!1_{\hat{\Omega}_\varepsilon}(x) + R_\varepsilon.
\end{aligned}$$

Since $\rho_\varepsilon \equiv 1$ in $\{x\in\Omega \,|\, \operatorname{dist}(x,\partial\Omega) > \varepsilon\}$ and since $\varepsilon|\nabla\rho_\varepsilon|$ is bounded in $L^\infty(\Omega)$, it is clear that the support of R_ε is included in $\{x\in\Omega \,|\, \operatorname{dist}(x,\partial\Omega) < 3\varepsilon\}\times Y$ and that R_ε is bounded in $L^\infty(\Omega\times Y)$. Thus $R_\varepsilon \to 0$ in $L^2(\Omega\times Y)$. It then suffices to note that $\nabla_y\hat{u}(y)1\!\!1_{\hat{\Omega}_\varepsilon}(x) \to \nabla_y\hat{u}(y)$ in $L^4(\Omega\times Y)$ in order to obtain the desired convergence result.

Similarly, we have $\mathcal{T}_\varepsilon(a_\varepsilon)(x,y) \to a(y)$ in $L^4(\Omega\times Y)$.

Finally,

$$\begin{aligned}
\limsup_\varepsilon \varepsilon^2 E_\varepsilon(U_\varepsilon) &\leq \lim_\varepsilon \varepsilon^2 E_\varepsilon(H_\varepsilon) = \lim_\varepsilon \frac{1}{2}\int_\Omega \left\{|\varepsilon\nabla H_\varepsilon|^2 + \frac{1}{2}(H_\varepsilon^2 - a_\varepsilon^2)^2\right\} \\
&= \left[\text{ with } \phi = |\varepsilon\nabla H_\varepsilon|^2 + \frac{1}{2}(H_\varepsilon^2 - a_\varepsilon^2)^2\right] \\
&= \lim_\varepsilon \frac{1}{2}\int_{\Omega\times Y} \mathcal{T}_\varepsilon(\phi) = [\text{here, we use u. c. i.}] \\
&= \lim_\varepsilon \frac{1}{2}\int_{\hat{\Omega}_\varepsilon\times Y} \left\{|\nabla\hat{u}|^2 + \frac{1}{2}(\hat{u}^2 - \mathcal{T}_\varepsilon(a_\varepsilon)^2)^2\right\} \\
&= \frac{1}{2}\int_{\Omega\times Y} \left\{|\nabla\hat{u}(y)|^2 + \frac{1}{2}(\hat{u}(y)^2 - a(y)^2)^2\right\} = |\Omega|\mathcal{E}(\hat{u}).
\end{aligned}$$

In order to complete the proof of Proposition 6, it suffices to establish the inequality

$$\liminf_\varepsilon \varepsilon^2 E_\varepsilon(U_\varepsilon) \geq |\Omega|\mathcal{E}(\hat{u}).$$

In order to obtain this estimate, we perform the following change of functions: for $u \in A := \{u \in H_1^1(\Omega)$ such that $b \leq u \leq 1\}$, we let $v := u^2$. We clearly have $v \in B := \{v \in H_1^1(\Omega)$ such that $b^2 \leq v \leq 1\}$. Both A and B are convex and closed in H_1^1. We have the following equivalences

$$\begin{aligned}
u \text{ minimizes } E_\varepsilon \text{ in } H_1^1(\Omega) &\Leftrightarrow u \text{ minimizes } E_\varepsilon \text{ in } \{u \in H_1^1(\Omega) \text{ such that } b \leq u \leq 1\} \\
&\Leftrightarrow u = \sqrt{v} \text{ minimizes } E_\varepsilon \text{ in } \{\sqrt{v} \in H_1^1(\Omega) \text{ such that } b^2 \leq v \leq 1\} \\
&\Leftrightarrow v = u^2 \text{ minimizes } G_\varepsilon \text{ in } \{v \in H_1^1(\Omega) \text{ such that } b^2 \leq v \leq 1\}
\end{aligned}$$

with

$$G_\varepsilon(v) := \frac{1}{4}\int_\Omega \left\{\frac{|\nabla v|^2}{2v} + \frac{1}{\varepsilon^2}(a_\varepsilon^2 - v)^2\right\}.$$

Let U_ε be the minimizer of E_ε in H_1^1. Then $V_\varepsilon := U_\varepsilon^2$ is the global minimizer of G_ε in $\{v \in H_1^1(\Omega)$ such that $b^2 \leq v \leq 1\}$. Let, for $v \in C := \{v \in H^1_{\mathrm{per}}(Y,\mathbb{R})$ such that $v \geq b^2\}$,

$$\mathcal{G}(v) := \frac{1}{4}\int_Y \left\{\frac{|\nabla v|^2}{2v} + (a^2 - v)^2\right\}.$$

It is clear that $\mathcal{G}$ has a unique minimizer in C, namely $\hat{v} := \hat{u}^2$.

With these notations, we have

$$\liminf_\varepsilon \varepsilon^2 E_\varepsilon(U_\varepsilon) = \liminf_\varepsilon \varepsilon^2 G_\varepsilon(V_\varepsilon) = \liminf_\varepsilon \frac{1}{4}\int_\Omega \left\{\frac{|\varepsilon\nabla V_\varepsilon|^2}{2V_\varepsilon} + (a_\varepsilon^2 - V_\varepsilon)^2\right\} = \liminf_\varepsilon \frac{1}{4}\int_\Omega \tilde{\phi}_\varepsilon(V_\varepsilon),$$

where $\tilde{\phi}_\varepsilon(V_\varepsilon) := \dfrac{|\varepsilon\nabla V_\varepsilon|^2}{2V_\varepsilon} + (a_\varepsilon^2 - V_\varepsilon)^2$. Using the bound $|\nabla U_\varepsilon| \leq \dfrac{C}{\varepsilon}$ [7], we see that $\displaystyle\int_{\Lambda_\varepsilon} \tilde{\phi}_\varepsilon(V_\varepsilon) \to 0$. This property, together with the properties i)-iv) of the unfolding operator, imply

$$\liminf_\varepsilon \int_\Omega \tilde{\phi}_\varepsilon(V_\varepsilon) = \liminf_\varepsilon \int_{\Omega\times Y} \mathcal{T}_\varepsilon(\tilde{\phi}(V_\varepsilon)), \tag{36}$$

where

$$\mathcal{T}_\varepsilon(\tilde{\phi}_\varepsilon(V_\varepsilon)) = \begin{cases} \dfrac{|\nabla_y \mathcal{T}_\varepsilon(V_\varepsilon)|^2}{2\mathcal{T}_\varepsilon(V_\varepsilon)} + (\mathcal{T}_\varepsilon(V_\varepsilon) - \mathcal{T}_\varepsilon(a_\varepsilon)^2)^2 & \text{in } \hat{\Omega}_\varepsilon \times Y \\ 0 & \text{in } \Lambda_\varepsilon \times Y \end{cases} := \phi^y_\varepsilon(\mathcal{T}_\varepsilon(V_\varepsilon)).$$

For $W \in L^2(\Omega, H^1(Y))$ such that $W \geq b^2$ a.e. in $\hat{\Omega}_\varepsilon \times Y$, define

$$\phi^y_\varepsilon(W) := \left(\frac{|\nabla_y W|^2}{2W} + (W - \mathcal{T}_\varepsilon(a_\varepsilon)^2)^2 \right) \mathbb{1}_{\hat{\Omega}_\varepsilon \times Y}.$$

Similarly, for $W \in L^2(\Omega, H^1(Y, \mathbb{R}))$ satisfying $W \geq b^2$ a.e. in $\Omega \times Y$, we denote

$$\phi^y(W) = \frac{|\nabla_y W(x,y)|^2}{2W(x,y)} + (W(x,y) - a(y)^2)^2.$$

One may prove that ϕ^y is a convex function of its argument W.

Using the strong convergence in $L^4(\Omega \times Y)$, as $\varepsilon \to 0$, of the family of $\mathcal{T}_\varepsilon(a_\varepsilon)$ to the map $(x,y) \mapsto a(y)$, it is not difficult to prove that the assumptions $W_\varepsilon \in L^2(\Omega, H^1(Y, \mathbb{R}))$, $W_\varepsilon \geq b^2$ a.e. in $\Omega \times Y$ and $|W_\varepsilon|, |\nabla_y W_\varepsilon| \leq C$ in $\Omega \times Y$ imply

$$\int_{\Omega \times Y} \{\phi^y_\varepsilon(W_\varepsilon) - \phi^y(W_\varepsilon)\} \to 0. \tag{37}$$

Since $\varepsilon \nabla V_\varepsilon$ is bounded in $L^\infty(\Omega)$ [7] and since V_ε is bounded in L^2, Corollary 3.2 in [12] implies that there exists some $\hat{V} \in L^2(\Omega, H^1_{\text{per}}(Y))$ such that, up to a subsequence, we have

$$\mathcal{T}_\varepsilon(V_\varepsilon) \rightharpoonup \hat{V} \text{ in } L^2(\Omega \times Y) \text{ and } \nabla_y(\mathcal{T}_\varepsilon(V_\varepsilon)) \rightharpoonup \nabla_y \hat{V} \text{ in } L^2(\Omega \times Y). \tag{38}$$

Let $W_\varepsilon := \mathcal{T}_\varepsilon(V_\varepsilon) + \mathbb{1}_{\Lambda_\varepsilon \times Y}$, which satisfies the assumptions leading to (37) and, in addition, satisfies

$$W_\varepsilon - \mathcal{T}_\varepsilon(V_\varepsilon) \to 0 \text{ in } L^2(\Omega \times Y) \text{ and } \nabla_y W_\varepsilon = \nabla_y \mathcal{T}_\varepsilon(V_\varepsilon).$$

To resume, the definition of W_ε combined with (38) yields

$$W_\varepsilon \rightharpoonup \hat{V} \text{ in } L^2(\Omega \times Y),\ \nabla_y W_\varepsilon \rightharpoonup \nabla_y \hat{V} \text{ in } L^2(\Omega \times Y),\ |W_\varepsilon|, |\nabla_y W_\varepsilon| \leq C \text{ and } W_\varepsilon \geq b^2 \tag{39}$$

(here, weak convergence is obtained after possibly passing to a subsequence.)

We are now in position to prove that $\liminf_\varepsilon \varepsilon^2 E_\varepsilon(U_\varepsilon) \geq |\Omega| \mathcal{E}(\hat{u})$. Indeed, using the fact that $\mathcal{T}_\varepsilon(a_\varepsilon) \to a$ in $L^4(\Omega \times Y)$ and the convexity of ϕ^y, we obtain

$$\begin{aligned} \liminf_\varepsilon \varepsilon^2 E_\varepsilon(U_\varepsilon) &= [\text{from } (36)] = \liminf_\varepsilon \frac{1}{4} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(\phi_\varepsilon(V_\varepsilon)) && (40) \\ &= [\text{since } W_\varepsilon = \mathcal{T}_\varepsilon(V_\varepsilon) \text{ in } \hat{\Omega}_\varepsilon \times Y] = \liminf_\varepsilon \frac{1}{4} \int_{\Omega \times Y} \phi^y_\varepsilon(W_\varepsilon) && (41) \\ &= [\text{using } (37), (39)] = \liminf_\varepsilon \frac{1}{4} \int_{\Omega \times Y} \phi^y(W_\varepsilon) && (42) \\ &\geq [\text{using } (39) \text{ and the convexity of } \phi^y] \geq \frac{1}{4} \int_{\Omega \times Y} \phi^y(\hat{V}) && (43) \\ &= \int_\Omega \mathcal{G}(\hat{V}(x,\cdot)) \mathrm{d}x \geq \int_\Omega \mathcal{G}(\hat{v}) \mathrm{d}x = |\Omega| \mathcal{E}(\hat{u}). && (44) \end{aligned}$$

It follows that

$$\lim_{\varepsilon} \varepsilon^2 E_\varepsilon(U_\varepsilon) = |\Omega|\mathcal{E}(\hat{u}).$$

The proof of Proposition 6 is complete. $\square$

We are now in position to complete the proof of Theorem 2, point *2.*, by identifying the weak limit of U_ε. From (40), it follows that, for a. e. $x \in \Omega$, $\hat{V}(x,\cdot)$ is a positive global minimizer of $\mathcal{G}$. For such x, we have $\hat{V}(x,\cdot) = \hat{v}(\cdot)$.

By combining the following facts:

$$\lim_{\varepsilon} \frac{1}{4}\int_{\Omega\times Y} \mathcal{T}_\varepsilon(\tilde{\phi}_\varepsilon(V_\varepsilon)) = |\Omega|\mathcal{E}(\hat{u}) = |\Omega|\mathcal{G}(\hat{v}),\ \mathcal{T}_\varepsilon(V_\varepsilon) \rightharpoonup \hat{v},\ \nabla_y \mathcal{T}_\varepsilon(V_\varepsilon) \rightharpoonup \nabla_y \hat{v} \text{ in } L^2(\Omega\times Y),$$

we obtain

$$\lim_{\varepsilon} \int_{\Omega\times Y} (\mathcal{T}_\varepsilon(V_\varepsilon) - \mathcal{T}_\varepsilon(a_\varepsilon)^2)^2 = \lim_{\varepsilon} \int_{\Omega\times Y} (\hat{v} - a^2)^2.$$

The above equality implies

$$\lim_{\varepsilon} \int_{\Omega\times Y} \mathcal{T}_\varepsilon(V_\varepsilon)^2 = \lim_{\varepsilon} \int_{\Omega\times Y} \hat{v}^2,$$

which in turn implies $\mathcal{T}_\varepsilon(V_\varepsilon) \to \hat{v}$ in $L^2(\Omega\times Y)$. Since $\hat{v} = \hat{u}^2$ and $V_\varepsilon = U_\varepsilon^2$, we obtain

$$\int_{\Omega\times Y} (\mathcal{T}_\varepsilon(U_\varepsilon) - \hat{u})^2 \le \frac{1}{4b^2}\int_{\Omega\times Y} (\mathcal{T}_\varepsilon(U_\varepsilon)^2 - \hat{u}^2)^2 \to 0,$$

that is, we find that $\mathcal{T}_\varepsilon(U_\varepsilon) \to \hat{u}$ in $L^2(\Omega\times Y)$. This fact combined with Proposition 2.9 iii) in [12] implies $U_\varepsilon \rightharpoonup \mathcal{M}_Y(\hat{u}) \equiv \int_Y \hat{u}(y)\mathrm{d}y$, which is the desired conclusion. $\square$

3.2.2 Limit of v_ε in H^1

Recall that we are in the critical case $\lambda = 1$, $\delta = \varepsilon$.

In order to state the main result of this section we recall the following standard existence result (see, e. g., Theorem 4.27 in [13])

Proposition 7. *Let $f \in (H^1_{\mathrm{per}}(Y))'$ have zero average. Then there exists an unique solution $h \in H^1_{per}(Y)$ of*

$$\mathrm{div}(\hat{u}^2\nabla h) = f \text{ and } \mathcal{M}_Y(h) = 0.$$

In view of this proposition, let $\chi_j \in H^1_{\mathrm{per}}(Y)$ be the unique solution of

$$\mathrm{div}(\hat{u}^2\nabla \chi_j) = \partial_j(\hat{u}^2) \text{ and } \mathcal{M}_Y(\chi_j) = 0. \tag{45}$$

Recall that the homogenized matrix $\mathcal{A}$ of $\hat{u}^2\left(\frac{x}{\varepsilon}\right)\mathrm{Id}_{\mathbb{R}^2}$ is given by

$$\mathcal{A} = \int_Y \hat{u}^2 \begin{pmatrix} 1-\partial_1\chi_1 & -\partial_1\chi_2 \\ -\partial_2\chi_1 & 1-\partial_2\chi_2 \end{pmatrix} \tag{46}$$

(see, e. g., [16] chapter 1 or [13] chapter 6).

Proposition 8. *Let φ_* be the unique solution of*

$$\begin{cases}\operatorname{div}(\mathcal{A}\nabla\varphi_*) = 0 & \text{in } \Omega\\ \varphi_* = \varphi_0 & \text{on } \partial\Omega\end{cases}. \tag{47}$$

Let $g = e^{\imath\varphi_0}$. Also, for small ε, represent a minimizer u_ε of E_ε in H^1_g as $u_\varepsilon = U_\varepsilon\rho_\varepsilon e^{\imath\varphi_\varepsilon}$, where $\varphi_\varepsilon \in H^1_{\varphi_0}(\Omega)$.

Then $\varphi_\varepsilon \rightharpoonup \varphi_*$ in $H^1(\Omega)$ as $\varepsilon \to 0$.

Proof. This argument is an adaptation of the proof of Theorem 4 in [22].

First note that $\mathcal{T}_\varepsilon(U_\varepsilon^2)(x,y) \to \hat{u}^2(y)$ in $L^2(\Omega\times Y)$ and $|v_\varepsilon|^2 = \rho_\varepsilon^2 \to 1$ in $L^2(\Omega)$ imply that

$$\mathcal{T}_\varepsilon(\rho_\varepsilon^2U_\varepsilon^2)(x,y) \to \hat{u}^2(y) \text{ in } L^2(\Omega\times Y).$$

Recalling that φ_ε is the solution of

$$\begin{cases}-\operatorname{div}(\rho_\varepsilon^2U_\varepsilon^2\nabla\varphi_\varepsilon) = 0 & \text{in } \Omega\\ \varphi_\varepsilon = \varphi_0 & \text{on } \partial\Omega\end{cases},$$

we find, using Proposition 6. iv) and the fact that $\rho_\varepsilon^2U_\varepsilon^2\nabla\varphi_\varepsilon \in H^1_{\text{loc}}(\Omega)$, that

$$0 = \varepsilon\mathcal{T}_\varepsilon\left(-\operatorname{div}(\rho_\varepsilon^2U_\varepsilon^2\nabla\varphi_\varepsilon)\right)(x,y) = -\operatorname{div}_y\left(\mathcal{T}_\varepsilon(\rho_\varepsilon^2U_\varepsilon^2)(x,y)\mathcal{T}_\varepsilon(\nabla\varphi_\varepsilon)(x,y)\right). \tag{48}$$

In order to prove that $\varphi_\varepsilon \rightharpoonup \varphi_*$ it suffices to prove that if, possibly up to a subsequence, we have $\varphi_\varepsilon \rightharpoonup \varphi^*$, then φ^* solves (47).

Using Theorem 3.5 in [12], we have the existence of $\hat{\varphi} \in L^2(\Omega, H^1_{\text{per}}(Y))$ such that

$$\mathcal{T}_\varepsilon(\nabla\varphi_\varepsilon) \rightharpoonup \nabla\varphi^* + \nabla_y\hat{\varphi} \text{ in } L^2(\Omega\times Y) \text{ and } \mathcal{M}_Y(\hat{\varphi}) = 0. \tag{49}$$

By inserting (49) into (48) and passing to the weak limits in $L^2(\Omega, H^{-1}(Y))$, we obtain

$$-\operatorname{div}_y\left[\hat{u}^2(y)\left(\nabla\varphi^*(x) + \nabla_y\hat{\varphi}(x,y)\right)\right] = 0$$

which is equivalent to

$$-\operatorname{div}_y\left[\hat{u}^2(y)\nabla_y\hat{\varphi}(x,y)\right] = \nabla_y\hat{u}^2(y)\cdot\nabla\varphi^*(x).$$

This equality combined with (45) implies that

$$\hat{\varphi}(x,y) = -\chi_1\partial_{x_1}\varphi^* - \chi_2\partial_{x_2}\varphi^*.$$

Consequently, we have

$$\nabla\varphi^* + \nabla_y\hat{\varphi} = \begin{pmatrix} 1-\partial_1\chi_1 & -\partial_1\chi_2\\ -\partial_2\chi_1 & 1-\partial_2\chi_2\end{pmatrix}\nabla\varphi^*.$$

On the other hand, let $\xi \in \mathcal{D}(\Omega)$. Then, for sufficiently small ε we have (*cf* Proposition 2.5. (i) in [12])

$$\begin{aligned}0 &= \lim_{\varepsilon\to0}\int_\Omega \rho_\varepsilon^2U_\varepsilon^2\nabla\varphi_\varepsilon\cdot\nabla\xi = \lim_{\varepsilon\to0}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(\rho_\varepsilon^2U_\varepsilon^2)\mathcal{T}_\varepsilon(\nabla\varphi_\varepsilon)\cdot\mathcal{T}_\varepsilon(\nabla\xi)\\ &= \int_\Omega\left\{\int_Y \hat{u}^2(y)(\nabla\varphi^* + \nabla_y\hat{\varphi})\right\}\cdot\nabla\xi = \int_\Omega \operatorname{div}_x\left\{\int_Y \hat{u}^2(y)(\nabla\varphi^* + \nabla_y\hat{\varphi})\right\}\xi.\end{aligned}$$

Therefore one has

$$\operatorname{div}_x\left[\int_Y \hat{u}^2(y)(\nabla\varphi^* + \nabla_y\hat{\varphi})\right] = \operatorname{div}_x\left[\int_Y \hat{u}^2(y)\begin{pmatrix} 1-\partial_1\chi_1 & -\partial_1\chi_2 \\ -\partial_2\chi_1 & 1-\partial_2\chi_2 \end{pmatrix}\nabla\varphi^*\right] = \operatorname{div}_x\left(\mathcal{A}\nabla\varphi^*\right) = 0$$

and consequently φ^* solves (47). $\square$

3.3 The case $\lambda = 1, \varepsilon \ll \delta$

Theorem 3. *Assume that $\lambda = 1$, $\delta \to 0$ and $\varepsilon/\delta \to 0$. Then, as $\varepsilon \to 0$, we have*

1. $\rho_\varepsilon = |u_\varepsilon| \rightharpoonup \mathcal{M}_Y(a)$ *in* $L^2(\Omega)$,

2. $\varphi_\varepsilon \rightharpoonup \varphi_*$ *in* $H^1(\Omega)$,

3. $\rho_\varepsilon^2\nabla\varphi_\varepsilon \rightharpoonup \mathcal{A}\nabla\varphi_*$ *in* $L^2(\Omega)$,

where φ_* solves the homogenized problem

$$\begin{cases} \operatorname{div}(\mathcal{A}\nabla\varphi_*) = 0 & \text{in } \Omega \\ \varphi_* = \varphi_0 & \text{on } \partial\Omega \end{cases}. \tag{50}$$

Here, $\mathcal{A}$ is the homogenized matrix of $a^2\left(\dfrac{x}{\delta}\right)\mathrm{Id}_{\mathbb{R}^2}$.

Proof. Theorem 1 combined with Lemma 5 yields $\rho_\varepsilon - a_\varepsilon \to 0$ in $L^2(\Omega)$. On the other hand, we have $a_\varepsilon \to \mathcal{M}_Y(a)$ weakly in $L^2(\Omega)$ (see, e. g.,[13] Theorem 2.6), so that *1.* follows.

In order to prove *2.* and *3.*, we start from the equation

$$\begin{cases} \operatorname{div}(\rho_\varepsilon^2\nabla\varphi_\varepsilon) = 0 & \text{in } \Omega \\ \varphi_\varepsilon = \varphi_0 & \text{on } \partial\Omega \end{cases} \tag{51}$$

satisfied by φ_ε. In view of the fact that $\rho_\varepsilon - a_\varepsilon \to 0$ in $L^2(\Omega)$, it is natural to compare φ_ε to the solution $\hat{\varphi}_\varepsilon$ of

$$\begin{cases} \operatorname{div}(a_\varepsilon^2\nabla\hat{\varphi}_\varepsilon) = 0 & \text{in } \Omega \\ \hat{\varphi}_\varepsilon = \varphi_0 & \text{on } \partial\Omega \end{cases}. \tag{52}$$

The difference $\psi_\varepsilon := \hat{\varphi}_\varepsilon - \varphi_\varepsilon$ is solution of

$$\begin{cases} \operatorname{div}(a_\varepsilon^2\nabla\psi_\varepsilon) = \operatorname{div}\left[(\rho_\varepsilon^2 - a_\varepsilon^2)\nabla\varphi_\varepsilon\right] & \text{in } \Omega \\ \psi_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{53}$$

We claim $\psi_\varepsilon \rightharpoonup 0$ in $H^1(\Omega)$. Indeed, we first note that, by (51), φ_ε is bounded in H^1. Using the fact that $b^2 \le a_\varepsilon^2 \le 1$ and (53) we obtain, via the Lax-Milgram theorem, that, with $C, C' > 0$ and $p < 2$ independent of ε, we have

$$\|\nabla\psi_\varepsilon\|_{L^2} \le C\|(\rho_\varepsilon^2 - a_\varepsilon^2)\nabla\varphi_\varepsilon\|_{L^2} \le C' < \infty$$

and (with $r := 2/(2-p)$)

$$\|\nabla\psi_\varepsilon\|_{L^p} \leq C\|(\rho_\varepsilon^2 - a_\varepsilon^2)\nabla\varphi_\varepsilon\|_{L^p} \leq C\|\rho_\varepsilon^2 - a_\varepsilon^2\|_{L^{rp}}\|\nabla\varphi_\varepsilon\|_{L^2}.$$

Consequently, ψ_ε is bounded in H_0^1 and converges strongly to 0 in $W^{1,p}(\Omega)$. It follows that $\psi_\varepsilon \rightharpoonup 0$ in $H^1(\Omega)$.

Now, using the classic periodic homogenization result (see, e. g., [16] chapter 1 or [13] chapter 6), we know that $\hat{\varphi}_\varepsilon \rightharpoonup \varphi_*$ in $H^1(\Omega)$ and $a_\varepsilon^2\nabla\hat{\varphi}_\varepsilon \rightharpoonup \mathcal{A}\nabla\varphi_*$ in $L^2(\Omega)$. These facts combined with the weak convergences $\psi_\varepsilon \rightharpoonup 0$ in $H^1(\Omega)$ and $(a_\varepsilon^2\nabla\hat{\varphi}_\varepsilon - \rho_\varepsilon^2\nabla\varphi_\varepsilon) \rightharpoonup 0$ in $L^2(\Omega)$ complete the proof of the theorem. □

3.4 The case $\lambda = 1, \delta \ll \varepsilon$

In this case, ε need not tend to 0. Up to subsequences, we may assume that either $\varepsilon = 1$ or $\varepsilon \to 0$.

Theorem 4. *The following hold.*

1. *Assume that $\varepsilon = 1$ and that $\delta \to 0$, and denote the energy by E_δ rather then E_ε. If u_δ is a minimizer of E_δ, then $u_\delta \rightharpoonup \hat{u}$ in $H^1(\Omega)$, where $\hat{u}$ solves*

$$\begin{cases} -\Delta\hat{u} = \hat{u}(\mathcal{M}_Y(a^2) - \hat{u}^2) & \text{in } \Omega \\ \hat{u} = g & \text{on } \partial\Omega \end{cases}. \tag{54}$$

2. *Assume that $\varepsilon \to 0$ and that $\delta/\varepsilon \to 0$. If $u_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}$ is a minimizer of E_ε, then we have*

   (i) $\rho_\varepsilon \to \sqrt{\mathcal{M}_Y(a^2)}$ strongly in $L^2(\Omega)$,

   (ii) $\varphi_\varepsilon \to \varphi_*$ in $H^1(\Omega)$.

   Here, φ_* denotes the harmonic extension of φ_0.

Proof. In case *1.*, we start by noting that $\|u_\delta\|_{H^1(\Omega)}$ is uniformly bounded with respect to δ. Let $\hat{u}$ be such that, possibly after passing to a subsequence, u_δ weakly converges to $\hat{u}$ in H^1. In order to identify $\hat{u}$, we let $\delta \to 0$ in the weak form of the GL equation satisfied by u_δ, namely:

$$\int_\Omega \nabla u_\delta \cdot \nabla\psi\, dx = \int_\Omega u_\delta(a_\delta^2 - u_\delta^2)\psi\, dx, \ \forall\ \psi \in C_0^\infty(\Omega)$$

and find that (54) holds.

In order to prove *2.*, we consider a partition of $\mathbb{R}^2$ by a family $\{C_k^\varepsilon\}$ of $\delta \times \delta$ squares. We may assume that

$$\{C_k^\varepsilon \,|\, C_k^\varepsilon \subset \{x \in \Omega \,|\, \mathrm{dist}(x, \partial\Omega) > \varepsilon\}\} = \{C_k^\varepsilon \,|\, k \in \{1, ..., N_\varepsilon\}\}.$$

Clearly, we have $N_\varepsilon = |\Omega|\delta^{-2} + \mathcal{O}(\varepsilon\delta^{-2})$. Denote $\Omega'_\varepsilon := \bigcup_{k=1}^{N_\varepsilon} C_k^\varepsilon$.

For $C_0 > 0$ (independent of ε) consider

$$\mathcal{H}_\varepsilon^{C_0} = \{w \in H^1_g \,|\, |\nabla w| \le \frac{C_0}{\varepsilon} \text{ in } \Omega'_\varepsilon \text{ and } |w| \le 1 \text{ in } \Omega\}.$$

Recall [7] that, for $\varepsilon < 1$ and a suitable C_0, each minimizer u_ε of E_ε in H^1_g belongs to $\mathcal{H}_\varepsilon^{C_0}$.

For $w \in \mathcal{H}_\varepsilon^{C_0}$, we have

$$\int_\Omega (|w|^2 - a_\varepsilon^2)^2 = \int_\Omega (|w|^2 - \mathcal{M}_Y(a^2))^2 + |\Omega| \left[\mathcal{M}_Y(a^4) - \mathcal{M}_Y(a^2)^2\right] + H_\varepsilon(w). \tag{55}$$

Here, the reminder H_ε satisfies $|H_\varepsilon(w)| \le o_\varepsilon(1)$, with $o_\varepsilon(1)$ independent of w. Indeed, we have

$$\begin{aligned}\int_\Omega (|w|^2 - a_\varepsilon^2)^2 - \int_\Omega (|w|^2 - \mathcal{M}_Y(a^2))^2 \;=\; & \int_\Omega \left[a_\varepsilon^2 - \mathcal{M}_Y(a^2)\right] a_\varepsilon^2 \\ & + \mathcal{M}_Y(a^2) \int_\Omega (a_\varepsilon^2 - \mathcal{M}_Y(a^2)) \\ & - 2 \int_\Omega \left[a_\varepsilon^2 - \mathcal{M}_Y(a^2)\right] |w|^2.\end{aligned}$$

We next note the three following facts. First, we have

$$\int_\Omega \left[a_\varepsilon^2 - \mathcal{M}_Y(a^2)\right] a_\varepsilon^2 = \sum_k \left\{\int_{C_\varepsilon^k} \left[a_\varepsilon^2 - \mathcal{M}_Y(a^2)\right] a_\varepsilon^2\right\} + \mathcal{O}(\varepsilon) = |\Omega| \left[\mathcal{M}_Y(a^4) - \mathcal{M}_Y(a^2)^2\right] + \mathcal{O}(\varepsilon).$$

Next, it holds that

$$\int_\Omega (a_\varepsilon^2 - \mathcal{M}_Y(a^2)) = \mathcal{O}(\varepsilon) + \sum_k \int_{C_\varepsilon^k} (a_\varepsilon^2 - \mathcal{M}_Y(a^2)) = \mathcal{O}(\varepsilon).$$

Finally, we have

$$\begin{aligned}\left|\int_\Omega \left[a_\varepsilon^2 - \mathcal{M}_Y(a^2)\right] |w|^2\right| \;\le\; & \mathcal{O}(\varepsilon) + \sum_k \int_{C_\varepsilon^k} \left|a_\varepsilon^2 - \mathcal{M}_Y(a^2)\right| |w|^2 \\ \le\; & \mathcal{O}(\varepsilon) + \sum_k \int_{C_\varepsilon^k} \left|a_\varepsilon^2 - \mathcal{M}_Y(a^2)\right| = o_\varepsilon(1).\end{aligned}$$

Thus (55) holds. Consequently, for $u \in \mathcal{H}_\varepsilon^{C_0}$, one has

$$E_\varepsilon(u) = \frac{|\Omega|}{4\varepsilon^2}(\mathcal{M}_Y(a^4) - \mathcal{M}_Y(a^2)^2) + G_\varepsilon(u) + o\left(\frac{1}{\varepsilon^2}\right), \tag{56}$$

where

$$G_\varepsilon(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (\mathcal{M}_Y(a^2) - |u|^2)^2.$$

We next claim that $\int_\Omega (|u_\varepsilon|^2 - \mathcal{M}_Y(a^2))^2\,dx \to 0$. Indeed, we consider a test function in the spirit of [18], more specifically we let $w_\varepsilon = |w_\varepsilon| e^{\imath\varphi_*}$, where φ_* is the harmonic extension of φ_0 and

$$|w_\varepsilon|(x) = \begin{cases} 1 - \dfrac{1-\sqrt{\mathcal{M}_Y(a^2)}}{\varepsilon}\mathrm{dist}(x,\partial\Omega), & \text{if } \mathrm{dist}(x,\partial\Omega) < \varepsilon \\ \sqrt{\mathcal{M}_Y(a^2)}, & \text{otherwise} \end{cases}.$$

Note that, for a suitable C_0, we have $w_\varepsilon \in \mathcal{H}_\varepsilon^{C_0}$. A straightforward computation yields $G_\varepsilon[w_\varepsilon] \le \dfrac{C}{\varepsilon}$.

Consequently, we obtain

$$E_\varepsilon(u_\varepsilon) \le E_\varepsilon(w_\varepsilon) \le \frac{|\Omega|}{4\varepsilon^2}(\mathcal{M}_Y(a^4) - \mathcal{M}_Y(a^2)^2) + o(\varepsilon^{-2}).$$

This estimate combined with (56) implies that $|u_\varepsilon| \to \sqrt{\mathcal{M}_Y(a^2)}$ strongly in $L^2(\Omega)$.

Using the second part of Corollary 1, we obtain that $\varphi_\varepsilon \to \varphi_*$ in $H^1(\Omega)$ where φ_* is the harmonic extension of φ_0.

The proof of Theorem 4 is complete. $\square$

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